Exercise Sheet 8

Discussion on 15.12.23

Exercise 1 (Conservation of Energy)
Prove that for the exact solution $u \in C^2([0,T] \times [0,1])$ to the wave equation, the energy

$$\frac{1}{2}\int_0^1 (\partial_t u(t,x))^2 + c^2(\partial_x u(t,x))^2 \, \mathrm{d}x$$

is constant in $t \in [0,T]$.

Exercise 2 (Discrete Maximum principle)
Consider the two-dimensional domain $[0,1]^2$. Let $J \in \mathbb{N}$ und $\Delta x = 1/J$. We consider a grid with grid points $x_{j,m} := (j\Delta x, m\Delta x)$. Define a discrete Laplace operator by

$$-\Delta_h U_{j,m} := -\partial_1^+ \partial_1^- U_{j,m} - \partial_2^+ \partial_2^- U_{j,m}.$$

Show the discrete maximum principle:
If $(U_{j,m})_{0\leq j,m\leq J}$ suffices $-\Delta_h U_{j,m} \leq 0$ for all $j,m = 1,\dots,J-1$ then U obtains its maximum at the boundary, i.e.

$$\max_{0\leq j,m\leq J} U_{j,m} \leq \max_{0\leq j\leq J} \max\{U_{0,j}, U_{J,j}, U_{j,0}, U_{j,J}\}.$$

Exercise 3
Let $x_{j,m}$ and $U_{j,m}$ as in exercise 2. Let $(U_{j,m})_{0\leq j,m\leq J}$ have zero boundary values, i.e.

$$U_{0,m} = U_{J,m} = U_{j,0} = U_{j,J} = 0 \qquad \text{für } 0 \leq j,m \leq J.$$

Then

$$\max_{0\leq j,m\leq J} |U_{j,m}| \leq \frac{1}{2} \max_{1\leq j,m\leq J-1} |\Delta_h U_{j,m}|. \tag{1}$$

Proceed as follows:

1. Define $W_{j,m} := (j\Delta x)^2 + (m\Delta x)^2$ and compute $\Delta_h W_{j,m}$.
2. Let $S := \max_{1\leq j,m\leq J-1} |\Delta_h U_{j,m}|$. Define $V_{j,m} := U_{j,m} + S\,W_{j,m}/4$. Use the discrete maximum principle from exercise 2 für $V_{j,m}$.
3. Conclude (1).

Exercise 4 (Programming Exercise)

1. Implement the stable naive implicit method for the wave equation given by the difference scheme

$$\partial_t^+ \partial_t^- U_j^k - c^2 \partial_x^+ \partial_x^- U_j^{k+1} = 0.$$

2. Implement the Crank-Nicolson-scheme for the wave equation given by the difference scheme

$$\partial_t^+ \partial_t^- U_j^k - \frac{c^2}{4} \partial_x^+ \partial_x^- (U_j^{k+1} + 2U_j^k + U_j^{k-1}) = 0.$$

3. Compare these both methods to the explicit method given in `wave_explicit.m` by some numerical experiments.

Hint: Both methods are implicit. The matrix corresponding to $\partial_x^+ \partial_x^- U_j^k$ can be used from the program `implicit_euler.m` from the website of the lecture.