Exercise Sheet 7

Discussion on 08.12.23

Exercise 1

(i) Show by constructing appropriate initial data that the difference scheme

$$U_j^{k+1} = U_j^k - \mu(U_j^k - U_{j-1}^k)$$

with $\mu = a\frac{\Delta t}{\Delta x}$ is unstable if $\mu > 1$.

(ii) Check the CFL condition and the estimate $\sup_{j=0,\dots,J} |U_j^{k+1}| \leq \sup_{j=0,\dots,J} |U_j^k|$ of the following difference schemes for the transport equation:

$$\partial_t^+ U_j^k + \partial_x^- U_j^k = 0$$
$$\partial_t^+ U_j^k + \partial_x^+ U_j^k = 0$$
$$\partial_t^+ U_j^k + \widehat{\partial}_x U_j^k = 0.$$

Exercise 2

Let $a < 0$ and consider the numerical scheme

$$\partial_t^+ U_j^k + a\partial_x^+ U_j^k = 0.$$

Show that the scheme is stable under appropriate conditions on Δt and Δx and prove an error estimate.

Exercise 3 (Discrete inverse inequality)

(a) Let $\Delta x > 0$ and $(V_j)_{j=0,\dots,J} \in \mathbb{R}^{J+1}$ with $V_0 = V_J = 0$. Prove that

$$\sum_{j=0}^{J-1} \Delta x \left(\frac{V_{j+1} - V_j}{\Delta x}\right)^2 \leq \frac{4}{(\Delta x)^2} \sum_{j=0}^{J} \Delta x V_j^2.$$

(b) Let $p \in P_k([a,b])$ be a polynomial of degree k on the interval $[a,b]$ with $a, b \in \mathbb{R}$ and $b > a$. Prove that for a constant $C > 0$ which is independent of a and b it holds

$$\|\partial_x p\|_{L^2([a,b])} \leq \frac{C}{b-a} \|p\|_{L^2([a,b])}.$$

Exercise 4 (Stability of Crank-Nicolson scheme)

Show that the Crank-Nicolson scheme is stable with respect to the supremum norm if $\lambda \leq 1$.