Exercise Sheet 6

Discussion on 01.12.23

Exercise 1 (Integration by parts)

a) Let $\Delta x > 0$, $(V_j)_{j=0,\dots,J} \in \mathbb{R}^{J+1}$, and $(W_j)_{j=0,\dots,J} \in \mathbb{R}^{J+1}$ with $V_0 = V_J = W_0 = W_J = 0$. Prove that

$$\sum_{j=1}^{J-1} \Delta x \left(\frac{V_{j+1} - 2V_j + V_{j-1}}{(\Delta x)^2} \right) W_j = - \sum_{j=0}^{J-1} \Delta x \left(\frac{V_{j+1} - V_j}{\Delta x} \right) \left(\frac{W_{j+1} - W_j}{\Delta x} \right).$$

b) Let $\Omega \subset \mathbb{R}^n, n = 1,2,3$ be a bounded domain with piecewise smooth boundary $\partial\Omega$, with outward normal ν along $\partial\Omega$. For $n = 1$, let $\operatorname{div} = \nabla$. For $v \in C^1(\overline{\Omega})$, $q \in C^1(\overline{\Omega}; \mathbb{R}^n)$, show

$$\int_\Omega (v \operatorname{div} q + \nabla v \cdot q) \, \mathrm{d}x = \int_{\partial\Omega} v q \cdot \nu \, \mathrm{d}s.$$

Hint: You may assume that Gauss's divergence theorem holds for bounded domains with piecewise smooth boundary.

Exercise 2 (Discrete version of Friedrichs inequality)

a) Let $J \in \mathbb{N}$, $J \geq 2$ and $A \in \mathbb{R}^{(J-1)\times(J-1)}$ given by

$$A = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{(J-1)\times(J-1)}.$$

Prove that for any $k = 1, \dots, J-1$, the vector $x^k \in \mathbb{R}^{J-1}$ with components $x^k_j = \sin(kj\pi/J)$ is an eigenvector of A with eigenvalue $\lambda_k := 2(1 - \cos(k\pi/J)) > 0$.

b) Show that $\pi^2/(2J^2) \leq \lambda_1$.

c) Use a), b) and the estimate $\lambda_{\min}(A)|v|^2 \leq v^\top A v$ for any $v \in \mathbb{R}^{J-1}$ to prove that there exists $C > 0$ such that any $J \in \mathbb{N}$, $\Delta x := 1/J$ and any $(V_j)_{j=0,\dots,J} \in \mathbb{R}^{J+1}$ with $V_0 = V_J = 0$ satisfy

$$\sum_{j=0}^{J-1} \Delta x \, V_j^2 \leq C \sum_{j=0}^{J-1} \Delta x \left(\frac{V_{j+1} - V_j}{\Delta x} \right)^2.$$

Exercise 3 (Semi-discrete heat equation)
Consider the semi-discrete PDE

$$\partial_t u - (\partial_x^+ \partial_x^-)u = 0,$$

where $u : [0,T] \to \mathbb{R}^{J-1}$ and $(\partial_x^+ \partial_x^-) : \mathbb{R}^{J-1} \to \mathbb{R}^{J-1}$ acts on vectors as $((\partial_x^+ \partial_x^-)u)_j = \partial_x^+ \partial_x^- u_j$ with the symmetric difference quotient from the lecture. For the definition of $\partial_x^+ \partial_x^- u_1$ and $\partial_x^+ \partial_x^- u_{J-1}$ we set $u_0 = u_J = 0$.

1. Compute a matrix $A \in \mathbb{R}^{(J-1)\times(J-1)}$ such that the above semi-discrete PDE is equivalent to the ODE $u'(t) = -Au(t)$.
2. Define $G : \mathbb{R}^{J-1} \to \mathbb{R}$ by $G(V) = \int_0^1 (v')^2 dx/(2\Delta x)$, where $v : [0,1] \to \mathbb{R}$ is defined by $v(0) = v(1) = 0$ and $v(j\Delta x) = V_j$ and a piecewise affine continuation between these values. Show that G satisfies $A = \nabla G$.
3. Conclude with Exercise 2(a) and Example 5.6 (1) from the lecture, that the explicit Euler scheme is stable, if $\tau \leq (\Delta x)^2/2$.

Exercise 4 (Error estimate for implicit Euler scheme)
Additionally to the notation of Proposition 6.2 from the lecture, let $T > 0$, $K \in \mathbb{N}$, $\Delta t := T/K$ and $t_k := k\Delta t$ for $k = 0, \ldots, K$. Prove that for $u \in C^4([0,T] \times [0,1])$ and $k = 0, \ldots, K$, the $(U_j^k)_{jk}$ from the implicit Euler scheme satisfy

$$\sup_{j=0,\ldots,J} |u(t_k, x_j) - U_j^k| \leq \frac{t_k}{2}(\Delta t + (\Delta x)^2)\big(\|\partial_x^4 u\|_{C([0,T]\times[0,1])} + \|\partial_t^2 u\|_{C([0,T]\times[0,1])}\big).$$