Exercise Sheet 1

Discussion on 27.10.23

Exercise 1
Prove theorem 2.3 from the lecture:
If $f \in \mathcal{C}^m([0,T]\times\mathbb{R}^n)$, *then* $y \in \mathcal{C}^{m+1}([0,T])$. *If* $m \geq 1$, *the solutions of the corresponding initial value problems are unique.*

Exercise 2 (Discrete Gronwall-Lemma)
Let $(u_k)_{k=0,\dots,K}$ be a sequence of non negative, real numbers and $\alpha, \beta \in \mathbb{R}$ with $\beta \geq 0$. Furthermore let

$$u_\ell \leq \alpha + \tau \sum_{k=0}^{\ell-1} \beta u_k \qquad \text{for all } \ell = 0, \dots, K.$$

Show that for all $\ell = 0, \dots, K$

$$u_\ell \leq \alpha e^{\ell\tau\beta}$$

holds. (*Hint: Use* $(1+\tau\beta) \leq e^{\tau\beta}$ *inductively.*)

Exercise 3 (Gerschgorin circle theorem)
Let $A \in \mathbb{R}^{n\times n}$ and let $\lambda \in \mathbb{C}$ be an eigenvalue of A. For $i \in \{1,\dots,n\}$ define the i-th Gerschgorin disc

$$D_i := \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{\substack{j=1,\dots,n \\ j\neq i}} |a_{ij}| \right\}.$$

Show that

$$\lambda \in \bigcup_{i=1}^{n} D_i.$$

Exercise 4
Let $D_1 = 1\frac{N}{m}, D_2 = 1.0201\frac{N}{m}$ be the spring constants of two spring pendulums with the same mass $m = 1kg$. The two pendulums are described by the following ODEs and their initial values

$$\begin{aligned} y_i''(t) &= -\frac{D_i}{m} y_i(t) && i \in \{1,2\} \text{ and } t \in [0,T] \\ y_i(0) &= 1 && i \in \{1,2\} \\ y_i'(0) &= 0 && i \in \{1,2\} \end{aligned}$$

Compute the unique solutions for y_1 and y_2 and their difference at time $T = 100\pi s \approx 314s$.