Exercise Sheet 8

Discussion on 16.12.2022

Exercise 1 (Dirichlet boundary data)
Consider the Poisson model problem with inhomogeneous Dirichlet boundary data: Given $u_D \in C(\Gamma_D)$, $g \in C(\Gamma_N)$ and $f \in L^2(\Omega)$, seek $u : \Omega \to \mathbb{R}$ with

$$-\Delta u = f \text{ in } \Omega, \quad u = u_D \text{ on } \Gamma_D, \quad \text{and} \quad \nabla u \cdot n = g \text{ on } \Gamma_N.$$

a) Modify the weak formulation of the Poisson model problem with the space $V_D := \{u \in H^1(\Omega) \mid u|_{\Gamma_D} = u_D\}$ to include inhomogeneous Dirichlet boundary data.
b) Utilize the split $u = u_0 + \tilde{u}_D$, where $u_0 \in H^1_D(\Omega)$ and $\tilde{u}_D \in V_D$ to incorporate the boundary data in the right-hand side of the formulation.

Exercise 2 (Error and refinement)
a) Let $(\mathcal{T}_k)_{k\in\mathbb{N}}$ be a sequence of regular triangulations, where $\mathcal{T}_{k+1}$ is a refinement of $\mathcal{T}_k$ for any $k \in \mathbb{N}$. Furthermore, let $u \in H^1_0(\Omega)$ be the exact solution and $u_k \in S^1_0(\Omega)$ the P_1 finite element solution to the Poisson model problem on each level $k \in \mathbb{N}$. Prove that $\|\nabla(u - u_k)\|_{L^2(\Omega)}$ is a monotonically decreasing sequence.
b) Consider the criss-cross triangulation $\mathcal{T}_0$ and its refinement depicted in Figure 1. Prove that the P_1 finite element solutions to the Poisson model problem with $f \equiv 1$ on the triangulations coincide.

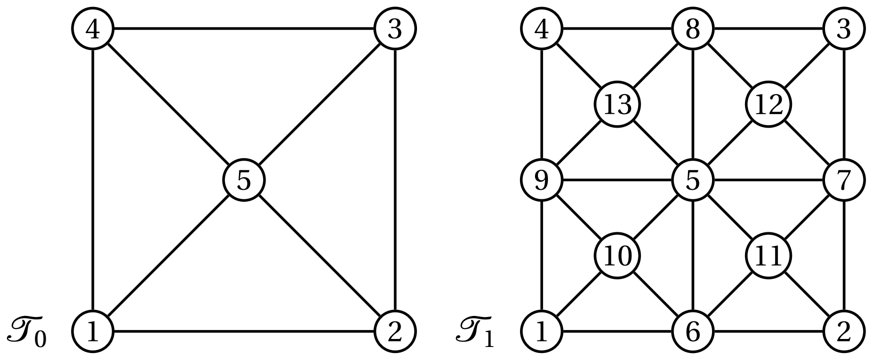

Figure 1: Criss-cross triangulation $\mathcal{T}_0$ (left) and its refinement $\mathcal{T}_1 = \texttt{bisec}(\texttt{bisec}(\mathcal{T}_0))$

Exercise 3 (1dFEM = Interpolation)
a) Let $\mathcal{T}$ be a triangulation of the Lipschitz domain Ω. For $f \in L^2(\Omega)$, define $\Pi_0 f \in P_0(\mathcal{T})$ by

$$\|f - \Pi_0 f\|_{L^2(\Omega)} = \min_{p_0 \in P_0(\mathcal{T})} \|f - p_0\|_{L^2(\Omega)}.$$

Prove that any $f \in L^2(\Omega)$ and $T \in \mathcal{T}$ satisfies

$$\Pi_0 f|_T = |T|^{-1} \int_T f \,\mathrm{d}x.$$

b) Let $0 = a_0 < a_1 < \ldots, a_N = 1$. Consider the 1d triangulation $\{(a_j, a_{j+1}) \mid j = 0, \ldots, N-1\}$. We consider the problem

$$-u'' = f \quad \text{auf } (0,1) \quad \text{und} \quad u(0) = 0 = u(b).$$

Use the L^2-projection from a) and Céas Lemma to prove that the nodal interpolation is the P_1-FEM solution in 1d.

Exercise 4 (Inf-sup condition for matrices)
Let U and V be finite-dimensional Hilbert spaces and $b : U \times V \to \mathbb{R}$ a bilinear form. Prove that the inf-sup constant

$$\alpha := \inf_{u \in U} \sup_{v \in V} \frac{b(u,v)}{\|u\|_U \|v\|_V}$$

corresponds to the smallest singular value of a matrix A representing the bilinar form b. Here, for fixed orthonormal bases $(\Phi_j)_{j=1,\ldots,m}$ of U and $(\Psi_k)_{k=1,\ldots,n}$ of V, the matrix A is defined via $A_{jk} = b(\Phi_j, \Psi_k)$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$.