Exercise Sheet 4

Exercise 1 (Euler formulas)
Let $\mathcal{T}$ denote a regular triangulation of the simply connected bounded domain Ω with nodes $\mathcal{N}$, edges $\mathcal{E}$ and interior edges $\mathcal{E}(\Omega)$. Prove that

$$|\mathcal{N}|+|\mathcal{T}| = 1+|\mathcal{E}|, \qquad 2|\mathcal{T}|+1 = |\mathcal{N}|+|\mathcal{E}(\Omega)|,$$

where $|\bullet|$ denotes the number of elements in a set. How can these formulas be generalized for multiply connected domains?

Exercise 2 (Minimum angle condition)
For any triangle T and node $z \in \mathcal{N}(T)$, denote by $\measuredangle(T,z)$ the interior angle of T at z. Prove that any family $(\mathcal{T}_k)_{k\in\mathbb{N}}$ of regular triangulations with

$$0 < \omega_0 \leq \min_{k\in\mathbb{N}} \min_{T\in\mathcal{T}_k} \min_{z\in\mathcal{N}(T)} \measuredangle(T,z) \tag{1}$$

is shape regular. Furthermore, find an example of a family of triangulations that does not satisfy (1) and is not shape regular.

Exercise 3 (Equivalences for shape regularity)
Consider a family of triangulations with the minimum angle condition (cf. Exercise 2) and let $\mathcal{T}$ be any triangulation of this family. Let $z \in \mathcal{N}$ and define $\mathcal{T}(z) := \{T \in \mathcal{T} \mid z \text{ is a node of } T\}$. Prove

$$\#\mathcal{T}(z) \lesssim 1.$$

Furthermore, for a triangle $T \in \mathcal{T}(z)$ with diameter h_T and an edge $E \in \mathcal{E}$ of T, prove

$$|E| \approx |T|^{1/2} \approx h_T,$$

where $|E|$ denotes the length of E and $|T|$ denotes the volume of T. Here, $A \lesssim B$ abbreviates, that there exists a constant $C < \infty$ that may depend on the minimum angle ω_0, but not on other properties of the family of triangulations. The formula $A \approx B$ abbreviates $A \lesssim B \lesssim A$.

Exercise 4 (Regularity of solutions)
Let $\gamma \in (\pi, 2\pi)$ and $\Omega := \{(r\cos\varphi, r\sin\varphi) \mid 0 < r < 1, 0 < \varphi < \gamma\}$ and $u(r,\varphi) = r^{\pi/\gamma}\sin(\varphi\pi/\gamma)$ in polar coordinates on Ω. Prove that $u : \Omega \to \mathbb{R}$ solves the Poisson problem $-\triangle u = 0$ on Ω with right-hand side $f \equiv 0$ and respective boundary data, but $u \notin C^1(\overline{\Omega})$ for $\gamma \in (\pi, 2\pi)$.
Hint: You may use the formula for the Laplacian in polar coordinates,

$$\Delta u = \partial^2 u/\partial r^2 + r^{-1}\partial u/\partial r + r^{-2}\partial^2 u/\partial\varphi^2.$$