Exercise Sheet 2

Discussion on 28.10.22

Exercise 1

(i) Show by constructing appropriate initial data that the difference scheme $U_j^{k+1} = U_j^k + \mu(U_j^k - U_{j-1}^k)$ with $\mu = a\frac{\triangle t}{\Delta x}$ is unstable if $\mu > 1$.

(ii) Check the CFL condition and the estimate $\sup_{j=0,\dots,J} |U_j^{k+1}| \leq \sup_{j=0,\dots,J} |U_j^k|$ of the following difference schemes for the transport equation:

$$\partial_t^+ U_j^k - \partial_x^- U_j^k = 0$$
$$\partial_t^+ U_j^k + \partial_x^+ U_j^k = 0$$
$$\partial_t^+ U_j^k + \widehat{\partial}_x U_j^k = 0.$$

Exercise 2

Let $a < 0$ and consider the numerical scheme $\partial_t^+ U_j^k + a\partial_x^- U_j^k = 0$. Show that the scheme is stable under appropriate conditions on $\triangle t$ and $\triangle x$ and prove an error estimate.

Exercise 3 (Discrete version of Friedrichs inequality)

a) Let $J \in \mathbb{N}$, $J \geq 2$ and $A \in \mathbb{R}^{(J-1)\times(J-1)}$ given by

$$A = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{(J-1)\times(J-1)}.$$

Prove that for any $k = 1, \dots, J-1$, the vector $x^k \in \mathbb{R}^{J-1}$ with components $x_j^k = \sin(kj\pi/J)$ is an eigenvector of A with eigenvalue $\lambda_k := 2(1 - \cos(k\pi/J)) > 0$.

b) Show that $\pi^2/(2J^2) \leq \lambda_1$.

c) Use a), b) and the estimate $\lambda_{\min}(A)|v|^2 \leq v^\top Av$ for any $v \in \mathbb{R}^{J-1}$ to prove that there exists $C > 0$ such that any $J \in \mathbb{N}$, $\Delta x := 1/J$ and any $(V_j)_{j=0,\dots,J} \in \mathbb{R}^{J+1}$ with $V_0 = V_J = 0$ satisfy

$$\sum_{j=0}^{J-1} \Delta x\, V_j^2 \leq C \sum_{j=0}^{J} \Delta x \left(\frac{V_{j+1} - V_j}{\Delta x}\right)^2.$$

Exercise 4 (Stability of Crank-Nicolson scheme)

Show that the Crank-Nicolson scheme is stable with respect to the supremum norm if $\lambda \leq 1$.