# Exercise Sheet 9

Discussion on 13.01.2023

**Exercise 1**

Let $\ell^2(\mathbb{N})$ be the space of all sequences $x = (x_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ such that $\|x\|^2_{\ell^2(\mathbb{N})} = \sum_{j\in\mathbb{N}} x_j^2 < \infty$. Show that the operator

$$L : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), (x_1, x_2, x_3, \ldots) \mapsto (x_1, x_2/2, x_3/3, \ldots)$$

is bounded, linear and injective, but Im$L$ is not closed.

**Exercise 2**

Let$\ell^2(\mathbb{N})$ be the space of all sequences $x = (x_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ such that $\|x\|_{\ell^2(\mathbb{N})^2} = \sum_{j\in\mathbb{N}} x_j^2 < \infty$. Determine which of the following operators

$$L_j : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), j = 1, 2, 3,$$

satisfies an inf-sup condition:

$$L_1 x = (x_1, 0, x_2, 0, x_3, 0, \ldots)$$
$$L_2 x = (x_1, x_3, x_3, \ldots)$$
$$L_3 x = (x_1 - x_2, x_3 - x_4, \ldots)$$

Specify for each operator its adjoint.

**Exercise 3**

Let $V$ be a Hilbert space and $a : V \times V \to \mathbb{R}$ be a bounded bilinear form that satisfies

$$\alpha_1 \|u\|_V \le \sup_{v\in V\setminus\{0\}} \frac{a(u,v)}{\|v\|_V} \qquad \text{for all } u \in V,$$
$$\alpha_2 \|v\|_V \le \sup_{u\in V\setminus\{0\}} \frac{a(u,v)}{\|u\|_V} \qquad \text{for all } v \in V.$$

1. Let $F \in V'$. Show that there exists a unique solution $u \in V$ to

$$a(u, v) = F(v) \qquad \text{for all } v \in V.$$

2. Does Brezzi's splitting lemma hold, if the coercivity of the bilinear form $a$ is replaced by the two inf-sup conditions from above?

3. Let $Q$ and $\Lambda$ be two Hilbert spaces and let $b : V \times Q \to \mathbb{R}$ and $c : Q \times \Lambda \to \mathbb{R}$ be two continuous bilinear forms. Define $Z(C) := \{q \in Q \mid \forall \mu \in \Lambda : c(q,\mu) = 0\} \subseteq Q$. Let $b$ and $c$ satisfy the inf-sup conditions

$$\beta \|q\|_Q \leq \sup_{v \in V \setminus \{0\}} \frac{b(v,q)}{\|v\|_V} \qquad \text{for all } q \in Z(C),$$
$$\gamma \|\mu\|_\Lambda \leq \sup_{q \in Q \setminus \{0\}} \frac{c(q,\mu)}{\|q\|_Q} \qquad \text{for all } \mu \in \Lambda.$$

Let $F \in V'$, $G \in Q'$ and $H \in \Lambda'$. Prove that there exists a unique solution $(u,p,\lambda) \in V \times Q \times \Lambda$ to

$$\begin{aligned} a(u,v) + b(v,p) \quad &= F(v) \qquad \text{for all } v \in V,\\ b(u,q) \quad + c(q,\lambda) &= G(q) \qquad \text{for all } q \in Q,\\ c(p,\mu) \quad &= H(\mu) \qquad \text{for all } \mu \in \Lambda. \end{aligned}$$

**Exercise 4 (Best-approximation with side condition)**
Suppose the closed subspaces $X_h$ and $M_h$ of the Hilbert spaces $X$ and $M$ satisfy the inf-sup condition and consider $g \in X^\star$ and the spaces

$$V(g) := \{v \in X \mid b(v,\mu) = \langle g,\mu\rangle_{X^\star,X} \text{ for any } \mu \in M\} \text{ and}$$
$$V_h(g) := \{v_h \in X_h \mid b(v_h,\mu_h) = \langle g,\mu_h\rangle_{X^\star,X} \text{ for any } \mu_h \in M_h\}.$$

Show that there exists $C > 0$ such that any $u \in V(g)$ satisfies

$$\inf_{v_h \in V_h(g)} \|u - v_h\|_X \leq C \inf_{w_h \in X_h} \|u - w_h\|_X.$$

*Hint:* Utilize the Fortin interpolation operator (Theorem 5.14).