Exercise Sheet 7

Discussion on 09.12.2022

Exercise 1 (Transformation of finite elements)
Let $(\widehat{T},\widehat{\mathscr{P}},\mathscr{K}_{\text{ref}})$ be a finite element and $\Phi_T : \widehat{T} \to T$ an affine diffeomorphism.
a) Show that $(T,\mathscr{P},\mathscr{K})$ is a finite element, where

$$T = \Phi_T(\widehat{T}),\quad \mathscr{P} = \{\widehat{q}\circ\Phi_T^{-1} \mid \widehat{q}\in\widehat{\mathscr{P}}\},\quad \mathscr{K} = \{\chi \mid \chi(v) = \widehat{\chi}(v\circ\Phi_T^{-1}) \text{ for all } v\in\mathscr{C}^\infty(T), \widehat{\chi}\in\widehat{\mathscr{K}}\}.$$

b) Show that the corresponding interpolants I_T and $I_{T_{\text{ref}}}$ and any $\widehat{v}\in W^{m,p}(T_{\text{ref}})$ and $v := \widehat{v}\circ\Phi_T^{-1}\in W^{m,p}(T)$ satisfy

$$(I_T v)\circ\Phi_T = I_{\widehat{T}}\widehat{v}.$$

Exercise 2 (Convergence history plots)
Suppose that the exact solution to the Poisson model problem u satisfies $u\in H^3(\Omega)$. Consider the sequence $(\mathscr{T}_k)_{k\in\mathbb{N}}$, where $\mathscr{T}_{k+1}$ is the red-refinement of $\mathscr{T}_k$ for any $k\in\mathbb{N}$. Furthermore, let $h_k := \max_{T\in\mathscr{T}_k} h_T$ and $\operatorname{ndof}(P_\ell,\mathscr{T}_k)$ the number of global degrees of freedom for the P_ℓ finite element method for $\ell = 1,2$ on $\mathscr{T}_k$ for $k\in\mathbb{N}$.
a) For a typical situation, sketch the functions $(h_k|u|_{H^2(\Omega)}, \operatorname{ndof}(P_\ell,\mathscr{T}_k))_{k\in\mathbb{N}}$ and $(h_k^2|u|_{H^3(\Omega)}, \operatorname{ndof}(P_\ell,\mathscr{T}_k))_{k\in\mathbb{N}}$ in a common log-log plot for $\ell = 1,2$.
(You may utilize the template in Figure 1.)
b) Where in these plots can the functions $(\|\nabla(u-u_k^{(1)})\|_{L^2(\Omega)}, \operatorname{ndof}(P_1,\mathscr{T}_k))_{k\in\mathbb{N}}$ and $(\|\nabla(u-u_k^{(2)})\|_{L^2(\Omega)}, \operatorname{ndof}(P_2,\mathscr{T}_k))_{k\in\mathbb{N}}$ lie for the P_1 (resp. P_2) finite element solutions $u_k^{(1)}$ (resp. $u_k^{(2)}$) on $\mathscr{T}_k$?

Exercise 3 (Approximation of the right hand side)
Es seien $u\in H_0^1(\Omega)$, $\widetilde{u}\in H_0^1(\Omega)$ und $\widehat{u}\in H_0^1(\Omega)$ Lösungen von

$$\begin{aligned}
\int_\Omega \nabla u\cdot\nabla v\,dx &= \int_\Omega f\,v\,dx &&\text{für alle } v\in H_0^1(\Omega),\\
\int_\Omega \nabla \widetilde{u}\cdot\nabla v\,dx &= \int_\Omega \Pi_0 f\,v\,dx &&\text{für alle } v\in H_0^1(\Omega),\\
\int_\Omega \nabla \widehat{u}\cdot\nabla v\,dx &= \sum_{T\in\mathscr{T}} f(x_T)\int_T v\,dx &&\text{für alle } v\in H_0^1(\Omega),
\end{aligned}$$

wobei $x_T\in T$ für alle $T\in\mathscr{T}$ ein beliebiger Punkt ist. Zeigen Sie, dass dann gilt, dass

$$\begin{aligned}
\|\nabla(u-\widetilde{u})\| &\lesssim \operatorname{osc}(f,\mathscr{T}),\\
\|\nabla(u-\widehat{u})\| &\lesssim \sum_{T\in\mathscr{T}} \|f - f(x_T)\|_{L^2(T)}.
\end{aligned}$$

Wenn $f \in H^1(\Omega)$, dann

$$\operatorname{osc}(f, \mathcal{T}) \lesssim \|\nabla f\| h^2,$$
$$\sum_{T \in \mathcal{T}} \|f - f(x_T)\|_{L^2(T)} \lesssim \|\nabla f\| h.$$

Ist $f \in H^2(\Omega)$ und ist $x_T = \operatorname{mid}(T)$ für alle $T \in \mathcal{T}$, dann gilt

$$\|\nabla(u - \widehat{u})\| \lesssim h^2.$$

Lässt sich die rechte Seite in der P^1-FEM wie für $\widehat{u}$ mit beliebigen $x_T \in T$ modifizieren, ohne dass die Konvergenzrate beeinträchtigt wird? Gilt dies auch für die P_2-FEM?

Exercise 4 (`FEM10`)
Study the MATLAB function `FEM10` below, which computes the P_1 finite element solution to the Poisson model problem for homogeneous Dirichlet boundary data and $f \equiv 1$. Modify it to include general right-hand sides $f \in L^2(\Omega)$, given as function handle.
Hint: You may use the midpoint quadrature rule from exercise 3 with $x_T = \operatorname{mid}(T)$ for integration

$$\int_T g(x)\, dx \approx |T| g(\operatorname{mid}(T)).$$

```
function [x,A,nrDoFs] = FEM10(c4n,n4e,n4sDb)
N=size(c4n,1); d=size(c4n,2);
A=sparse(N,N); b=zeros(N,1); x=zeros(N,1);
for j=1:size(n4e,1)
  area=abs(det([ones(1,d+1);c4n(n4e(j,:),:)']))/factorial(d));
  grads=[ones(1,d+1);c4n(n4e(j,:),:)']\[zeros(1,d);eye(d)];
  A(n4e(j,:),n4e(j,:))=A(n4e(j,:),n4e(j,:))+area*(grads*grads');
  b(n4e(j,:))=b(n4e(j,:))+ones(d+1,1)*area/(d+1); end
freeNodes=setdiff(1:N,unique(n4sDb)); nrDoFs=length(freeNodes);
x(freeNodes)=A(freeNodes,freeNodes)\b(freeNodes); end
```

You can test your code with the FEM10 program from the lecture, which you can find on the webpage of the lecture.

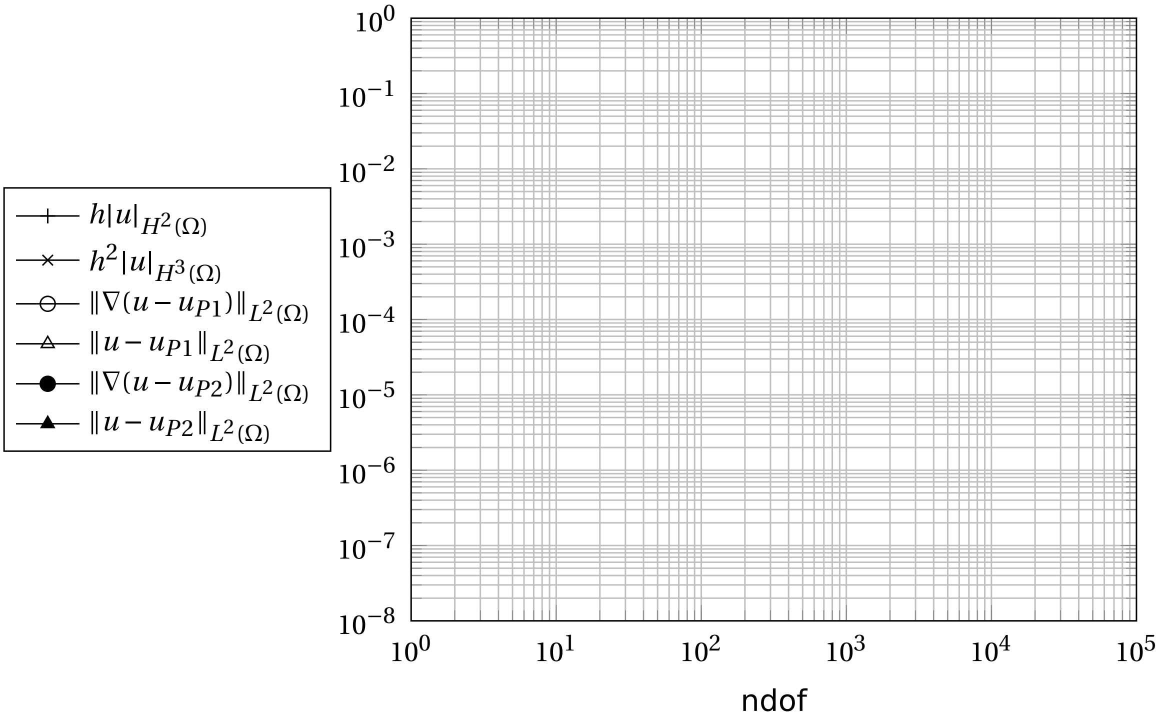

Figure 1: Model convergence history plot for Exercise 1