Exercise Sheet 6

Discussion on 02.12.2022

Exercise 1 (Q_k-FEM)
Let $T = [0,1]^2 \subseteq \mathbb{R}^2$, $k \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_k = 1$ and define $Q_{ij} := (t_i, t_j) \in T$. Furthermore, let $\mathcal{K} = \{\chi_{ij} \mid i, j = 0, \ldots, k\}$ with $\chi_{ij}(v) := v(Q_{ij})$ for $v \in C^\infty(T)$. Prove that $(T, Q_k(T), \mathcal{K})$ is a finite element in the sense of Ciarlet (where $Q_k(T)$ is the space of polynomials of partial degree $\le k$).

Exercise 2
Let $T \subseteq \mathbb{R}^2$ be a triangle and let $0 < t < 1/2$. Let $E = \operatorname{conv}\{a, b\} \subseteq T$ be an edge of T with midpoint $\operatorname{mid}(E) = (a+b)/2$ and define

$$\begin{aligned} p_E^\pm &:= \operatorname{mid}(E) \pm t(a-b) \in E, \\ \chi_{p_E^\pm}(v) &:= v(p_E^\pm) \qquad\qquad \text{for all } v \in \mathscr{C}^\infty(T). \end{aligned}$$

Define $\mathcal{K} := \{\chi_{p_E^\pm} \mid E \text{ is an edge of } T\}$. Show that $(T, P_2(T), \mathcal{K})$ is *not* a finite element in the sense of Ciarlet.

Exercise 3 (Barycentric coordinates)
Consider a triangle $T = \operatorname{conv}\{P_1, P_2, P_3\}$ and the barycentric coordinates $\lambda_1, \lambda_2, \lambda_3 \in P_1(T)$ defined via $\lambda_j(P_k) = \delta_{jk}$ for $j, k = 1, 2, 3$.
a) Prove that any $\alpha, \beta, \gamma \in \mathbb{N}_0$ satisfy

$$\int_T \lambda_1^\alpha \lambda_2^\beta \lambda_2^\gamma \,\mathrm{d}x = 2|T| \frac{\alpha!\beta!\gamma!}{(2+\alpha+\beta+\gamma)!}.$$

b) For the points $P_{j+3} := (P_j + P_{j+1})/2$, $j = 1, 2$ and $P_6 := (P_3 + P_1)/2$, find the nodal basis functions $\mu_j \in P_2(T)$ with $\mu_j(P_k) = \delta_{jk}$ for $j, k = 1, \ldots, 6$.
c) Compute the local mass matrices for $P_1(T)$ and $P_2(T)$, i.e. $M_1 = ((\lambda_i, \lambda_j)_{L^2(\Omega)})_{i,j=1,\ldots,3} \in \mathbb{R}^{3\times 3}$ and $M_2 = ((\mu_i, \mu_j)_{L^2(\Omega)})_{i,j=1,\ldots,6} \in \mathbb{R}^{6\times 6}$.

Exercise 4 (P_2 is no C^1 element)
Consider a regular triangulation $\mathcal{T}$ and the P_2 finite element $(T, P_2(T), \mathcal{K}_T)$ for any $T \in \mathcal{T}$. Prove that this finite element is not a C^1 finite element in general.