Exercise Sheet 5

Discussion on 18.11.2022

Exercise 1 (Weak derivatives are necessary)
Let $\Omega = (-1,1)^n \subseteq \mathbb{R}^n$ and $v_\varepsilon, \operatorname{sgn} : \overline{\Omega} \to \mathbb{R}$ with $v_\varepsilon(x) := \sqrt{|x|^2 + \varepsilon^2} - \varepsilon$ and

$$\operatorname{sgn}(x) := \begin{cases} \frac{x}{|x|} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Prove that v_ε converges to $v = |\bullet|$ and the gradient of v_ε converges to sgn in $L^2(\Omega)$ for $\varepsilon \searrow 0$. Use this to show that $C^1(\overline{\Omega})$, equipped with the scalar product $(\bullet, \bullet)_{L^2(\Omega)} + (\nabla \bullet, \nabla \bullet)_{L^2(\Omega)}$, is *not* a Hilbert space.

Exercise 2 (Energy minimization)
Let V be a Hilbert space with scalar product $(\bullet, \bullet)_V$, $a : V \times V \to \mathbb{R}$ a symmetric bilinear form with $0 \leq a(v,v)$ for any $v \in V$. Given $f \in V$, define $G(v) := \frac{1}{2} a(v,v) - (f,v)_V$ for any $v \in V$. Prove that the following two statements are equivalent for $u \in V$:

1. $G(u) = \min_{v \in V} G(v)$,
2. $a(u,v) = (f,v)_V$ for any $v \in V$.

Exercise 3 (Globally continuous, piecewise differentiable functions)
Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain and let $(\Omega_j)_{j=1,\ldots,J}$ be Lipschitz domains that are disjoint subsets of Ω with $\overline{\Omega} = \overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_J$. Let $u : \Omega \to \mathbb{R}$ with $u|_{\Omega_j} \in C^1(\overline{\Omega}_j)$ for all $j = 1, \ldots, J$. Show that u is weakly differentiable if and only if $u \in C(\overline{\Omega})$.

Exercise 4 (Solutions to PMP)
In the setting of the Poisson model problem, let $V = H^1_D(\Omega)$, $a : V \times V \to \mathbb{R}$ defined by $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$ and $F = \int_\Omega f v \, \mathrm{d}x + \int_{\Gamma_N} g v \, \mathrm{d}s$. Prove that there exists a unique solution $u \in V$ to

$$a(u,v) = F(v) \quad \text{for any } v \in V,$$

and that this solution satisfies

$$\|u\|_{H^1[\Omega)} \leq (1 + C_P^2) \max\{1, C_\gamma\} \big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_N)} \big).$$