Exercise Sheet 3

Exercise 1 (Conservation of Energy)
Prove that for the exact solution $u \in C^2([0,T] \times [0,1])$ to the wave equation, the energy

$$\frac{1}{2}\int_0^1 (\partial_t u(t,x))^2 + c^2(\partial_x u(t,x))^2 \,\mathrm{d}x$$

is constant in $t \in [0,T]$.

Exercise 2 (Discrete Maximum principle)
Consider the two-dimensional domain $[0,1]^2$. Let $J \in \mathbb{N}$ und $\Delta x = 1/J$. We consider a grid with grid points $x_{j,m} := (j\Delta x, m\Delta x)$. Define a discrete Laplace operator by

$$-\Delta_h U_{j,m} := -\partial_1^+\partial_1^- U_{j,m} - \partial_2^+\partial_2^- U_{j,m}.$$

Show the discrete maximum principle. If $(U_{j,m})_{0\le j,m\le J}$ suffices $-\Delta_h U_{j,m} \le 0$ for all $j,m = 1,\ldots,J-1$ then U obtains its maximum at the boundary, i.e.

$$\max_{0\le j,m\le J} U_{j,m} \le \max_{0\le j\le J} \max\{U_{0,j}, U_{J,j}, U_{j,0}, U_{j,J}\}.$$

Exercise 3
Let $x_{j,m}$ and $U_{j,m}$ as in exercise 2. Let $(U_{j,m})_{0\le j,m\le J}$ have zero boundary values, i.e.

$$U_{0,m} = U_{J,m} = U_{j,0} = U_{j,J} = 0 \qquad \text{für } 0 \le j,m \le J.$$

Then

$$\max_{0\le j,m\le J} |U_{j,m}| \le \frac{1}{2} \max_{1\le j,m\le J-1} |\Delta_h U_{j,m}|. \tag{1}$$

Proceed as follows:

1. Define $W_{j,m} := (j\Delta x)^2 + (m\Delta x)^2$ and compute $\Delta_h W_{j,m}$.
2. Let $S := \max_{1\le j,m\le J-1} |\Delta_h U_{j,m}|$. Define $V_{j,m} := U_{j,m} + S\,W_{j,m}/4$. Use the discrete maximum principle from exercise 2 für $V_{j,m}$.
3. Conclude (1).

Exercise 4 (Finite-Differences for the Poisson problem (five-point stencil))
Let $\Omega = (0,1)^2$, $\Gamma_D = \partial\Omega$ and $u_D = 0$. Let u denote the solution to the Poisson problem

$$-\Delta u = f \qquad \text{in } \Omega,$$
$$u|_{\Gamma_D} = 0.$$

Let $x_{j,m} := (j\Delta x, m\Delta x)$ be as in exercise 2 and 3. The finite difference approximation $(U_{j,m})_{0\leq j,m\leq J}$ is defined by

$$
\begin{aligned}
-\Delta_h U_{j,m} = f(x_{j,m}) \qquad & \text{für } 1 \leq j, m \leq J-1, \\
U_{0,m} = U_{J,m} = U_{j,0} = U_{j,J} = 0 \qquad & \text{for } 0 \leq j, m \leq J.
\end{aligned}
$$

Let $u \in \mathscr{C}^4(\overline{\Omega})$. Use exercise 3, to prove the error estimation

$$\sup_{0\leq j,m\leq J} |u(x_{j,m} - U_{j,m})| \leq \frac{\Delta x^2}{12} \|u\|_{\mathscr{C}^4([0,1]^2)}.$$