Exercise Sheet 13

Exercise 1 (arbitrary oscillations for efficiency)
Prove that the efficiency estimate from theorem 6.5 also holds for oscillations of higher order, i.e. for oscillations

$$\operatorname{osc}_k(f,\mathcal{T}) := \|h_{\mathcal{T}}(f-\Pi_k f)\|,$$

where $\Pi_k : L^2(\Omega) \to P_k(\mathcal{T})$ denotes the L^2-projection onto piecewise polynomials of degree smaller or equal to k.

Exercise 2 (Arbitrarily bad convergence for uniform refinements)
Let $\mathcal{T}_\ell, \ell \in \mathbb{N}$, denote a sequence of consecutive red-refinements of the initial regular triangulation $\mathcal{T}_0$ of Ω. Given an arbitrary monotonically decreasing sequence $\varepsilon_k, k \in \mathbb{N}$, show that there exists a right-hand side $f \in H^{-1}(\Omega) := (H_0^1(\Omega))^*$ such that the exact weak solution $u \in H_0^1(\Omega)$ to the Poisson model problem $-\Delta u = f$ and the corresponding finite element approximations $u_\ell \in S_0^1(\mathcal{T}_\ell)$ satisfy

$$\|\nabla(u-u_\ell)\|_{L^2(\Omega)} = \varepsilon_\ell.$$

Exercise 3 (Convergence for any refinement)
Let Ω denote a polygonal Lipschitz domain, $f \in L^2(\Omega)$, $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ a scalar product and $\mathcal{T}_0$ an initial regular triangulation of Ω. Let $\mathcal{T}_\ell, \ell \in \mathbb{N}$, be a sequence of arbitrary consecutive refinements, i.e., $\mathcal{T}_\ell$ is an (arbitrary) refinement of $\mathcal{T}_{\ell-1}$. Let $u_\ell \in S_0^1(\mathcal{T}_\ell) \subset H_0^1(\Omega)$ be the associated finite element solutions, i.e.,

$$a(u_\ell, v_\ell) = \int_\Omega f v_\ell \, dx \qquad \text{for all } v_\ell \in S_0^1(\mathcal{T}_\ell).$$

(a) Prove that the sequence $u_\ell, \ell \in \mathbb{N}$, converges to a function $u_\infty \in H_0^1(\Omega)$.

(b) Does the limit u_∞ solve the weak formulation

$$a(u,v) = \int_\Omega f v \, dx \qquad \text{for all } v \in H_0^1(\Omega)\,? \tag{1}$$

(c) State a sufficient criterion on the sequence $\mathcal{T}_\ell, \ell \in \mathbb{N}$, such that $u_\infty = u$ holds with the solution $u \in H_0^1(\Omega)$ to the weak formulation (1).

Exercise 4
Show that the solution to P_1-FEM in general does not converge better than with rate h.