Exercise Sheet 12

Exercise 1 (Instable P_1-P_0 element for the Stokes equations)
Consider the Stokes equations on the domain $\Omega := (0,1)^2$. Prove that the discretization $X_h := S_0^1(\mathcal{T};\mathbb{R}^2)$ and $Y_h := P_0(\mathcal{T}) \cap L_0^2(\Omega)$ for a criss triangulation $\mathcal{T}$ of Ω (i.e. a set of congruent squares which are bisected from the lower left to the upper right corner, see Figure 1) lead to an unstable saddle point problem.

Hint: Therefore show that the inf-sup condition is not satisfied or show that $\{v_h \in X_h \mid \forall q_h \in Y_h : \int_\Omega q_h \operatorname{div} v_h \,\mathrm{d}x = 0\} = \{0\}$.

Exercise 2 (Fortin operator for P_2-P_0 element for the Stokes equations – Compatibility)
Recall the averaging operator $J_1 : P_1(\mathcal{T}) \to S_0^1(\mathcal{T})$ with

$$(J_1 g)(z) := \sum_{T\in\mathcal{T}(z)} (g|_T)(z) \big/ \#(\mathcal{T}(z)) \quad \text{for all } z \in \mathcal{N}(\Omega) \text{ with } \mathcal{T}(z) := \{T \in \mathcal{T} \mid z \in T\}.$$

For any interior edge $E \in \mathcal{E}(\Omega)$, define the P_2 basis function $\varphi_E \in S_0^2(\mathcal{T})$ on E by piecewise quadratic interpolation of the values

$$\varphi_E(z) = 0 = \varphi_E(\operatorname{mid}(F)) \quad \text{for all } z \in \mathcal{N} \text{ and } F \in \mathcal{E}(\Omega) \setminus \{E\} \quad \text{and} \quad \varphi_E(\operatorname{mid}(E)) = 1.$$

Given $v \in H_0^1(\Omega;\mathbb{R}^2)$ and some coefficients $\{\alpha_E \in \mathbb{R}^2 \mid E \in \mathcal{E}(\Omega)\}$, set

$$v_h := J_1(\Pi_0 v) + \sum_{E\in\mathcal{E}(\Omega)} \alpha_E \varphi_E \in S_0^2(\mathcal{T};\mathbb{R}^2).$$

Show, that there exists a specific choice of the coefficients α_E such that

$$\int_\Omega q_h \operatorname{div}(v - v_h) \,\mathrm{d}x = 0 \quad \text{for all } q_h \in P_0(\mathcal{T}).$$

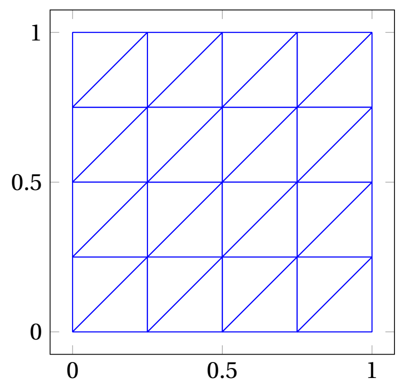

Figure 1: Example of a criss triangulation

Exercise 3 (Fortin operator for P_2-P_0 element for the Stokes equations – Stability)
Prove that there exists an h-independent generic constant $C > 0$ such that, for any $v \in H_0^1(\Omega;\mathbb{R}^2)$, the construction $v_h \in S_0^2(\mathcal{T};\mathbb{R}^2)$ from Exercise 2 satisfies

$$\|\nabla v_h\|_{L^2(\Omega)} \leq C\|\nabla v\|_{L^2(\Omega)}.$$

Exercise 4 (L^2-estimation for the Stokes problem)
Let $||v||_k := ||v||_{H^k(\Omega)}$ and $||v||_0 := ||v||_{L^2(\Omega)}$. For the Stokes problem on a convex domain with suffiently smooth boundary the folling statement on the regularity holds: (compare Girault and Raviart [1986])

$$||u||_2 + ||p||_1 \leq c||f||_0$$

Show with the usual duality technique that the L^2-estimation for the velocity

$$||u - u_h||_0 \leq ch(||u - u_h||_1 + ||p - p_h||_0)$$

holds.