Exercise Sheet 11

discussion on 20.01.2023

Exercise 1 (Korn inequality)
Let $\Omega \subseteq \mathbb{R}^n$ denote a Lipschitz domain. Recall the definition of the symmetric gradient $\varepsilon(v) := (\mathrm{D}\, v + \mathrm{D}\, v^\top)/2$ for all $v \in H^1(\Omega;\mathbb{R}^n)$.

(a) Use integration by parts to prove for all $v \in \mathscr{D}(\Omega;\mathbb{R}^n) \equiv C_c^\infty(\Omega;\mathbb{R}^n)$, that

$$\|\mathrm{D}\, v\|_{L^2(\Omega)} \lesssim \|\varepsilon(v)\|_{L^2(\Omega)}. \tag{1}$$

(b) Use a density argument to prove equation (1) for all $v \in H_0^1(\Omega;\mathbb{R}^n)$.

Exercise 2 (Variational formulation of linear elasticity)
Given a right-hand side f and Lamé parameters $\lambda, \mu > 0$, the PDEs of linear elasticity with homogeneous Dirichlet boundary conditions seek $u \in C^2(\Omega;\mathbb{R}^n)$ with

$$-\operatorname{div}(\mathbb{C}\varepsilon(u)) = f \text{ in } \Omega \quad \text{and} \quad u \equiv 0 \text{ on } \partial\Omega, \tag{2}$$

where the material tensor $\mathbb{C}$ is defined by $\mathbb{C}M := 2\mu M + \lambda \operatorname{tr}(M) I_{n\times n}$ for $M \in \mathbb{R}^{n\times n}$.

(a) Derive the weak formulation of (2).

(b) Prove existence and uniqueness of solutions of the variational formulation.
Hint: Assume that the space $H_0^1(\Omega;\mathbb{R}^n)$ of the deformations u is equipped with the usual H^1 energy-norm.

Exercise 3 (Conforming P_1 discretization)
Let $\mathscr{T}$ denote a regular triangulation of the domain Ω. Consider the conforming P_1 discretization $u_h \in S_0^1(\mathscr{T};\mathbb{R}^n)$ of the deformation $u \in H_0^1(\Omega;\mathbb{R}^n)$ in the weak formulation of Exercise 2 (a). Use Céa's Lemma to prove a best-approximation result in the H^1 (semi-) norm. How does the generic constant in the estimate depend on the Lamé parameter λ for large λ?

Exercise 4
Show that the linear elasticity problem is equivalent to a saddle point problem with a penalty term of the form

$$\begin{aligned} a(u,v) + b(v,p) &= f && \text{für alle } v \in H_0^1(\Omega),\\ b(u,q) + c(p,q) &= 0 && \text{für alle } q \in L^2(\Omega), \end{aligned}$$

where $p = \lambda \operatorname{div} u$