Exercise Sheet 10

discussion on 13.01.23

Exercise 1 (Gradients and divergence)
a) Suppose that $u \in L^2(\Omega)$ and $p \in L^2(\Omega;\mathbb{R}^n)$. Show that $p = -\nabla u$ and $u \in H_0^1(\Omega)$ is equivalent to

$$\int_\Omega u \operatorname{div} q \, \mathrm{d}x = \int_\Omega p \cdot q \, \mathrm{d}x \quad \text{for any } q \in H(\operatorname{div},\Omega).$$

b) Suppose that $g \in L^2(\Omega)$ and $p \in L^2(\Omega;\mathbb{R}^n)$. Show that $g = -\operatorname{div} p$ and $p \in H(\operatorname{div},\Omega)$ with $p \cdot \nu = 0$ on $\partial\Omega$ is equivalent to

$$\int_\Omega vg \, \mathrm{d}x = \int_\Omega p \cdot \nabla v \, \mathrm{d}x \quad \text{for any } v \in H^1(\Omega).$$

Exercise 2 ($H(\operatorname{div},\Omega)$ functions do not have normal jumps)
a) Consider the space

$$H(\operatorname{div},\Omega) := \left\{ v \in L^2(\Omega;\mathbb{R}^2) \;\middle|\; \exists g \in L^2(\Omega) \text{ with } \int_\Omega v \cdot \nabla\varphi \, dx = \int_\Omega g\varphi \, dx \text{ for all } \varphi \in \mathscr{C}_c^\infty(\Omega). \right\}$$

Let $u \in H^1(\Omega)$ be the exact weak solution of the Poisson problem $-\triangle u = f$ with some $f \in L^2(\Omega)$. Prove that $\nabla u \in H(\operatorname{div},\Omega)$.
b) Let $\mathscr{T}$ be a regular triangulation of $\Omega \subseteq \mathbb{R}^2$ and
$q \in H^1(\mathscr{T};\mathbb{R}^2) := \{q \in L^2(\Omega;\mathbb{R}^2) \mid q|_T \in H^1(T) \text{ for any } T \in \mathscr{T}\}$.
Prove that $q \in H(\operatorname{div},\Omega)$ if and only if for any $E = T_+ \cap T_-$ with $T_+, T_- \in \mathscr{T}$, $[q]_E \cdot \nu_E := (q_{T_+} \cdot \nu_{T_+} + q_{T_-} \cdot \nu_{T_-})|_E = 0$ almost everywhere along E.
c) Let $f \in L^2(\Omega) \setminus \{0\}$ and $u \in H_0^1(\Omega)$ the exact weak solution to $-\Delta u = f$. For an arbitrary regular triangulation $\mathscr{T}$ of Ω, let $u_h \in S_0^1(\mathscr{T})$ be the P_1 finite element solution. Prove that $u_h \neq u$.

Exercise 3 (Trace identity and inequality)
a) Suppose $T := \operatorname{conv}(\{P\} \cup E) \subset \mathbb{R}^2$ is a triangle with node P and opposite edge E. Prove that any $f \in H^1(T)$ satisfies the trace identity

$$|E|^{-1} \int_E f \, \mathrm{d}s = |T|^{-1} \int_T f \, \mathrm{d}x + \frac{1}{2} |T|^{-1} \int_T \nabla f(x) \cdot (x - P) \, \mathrm{d}x.$$

b) In the setting of a), show that $f \in H^1(T)$ and $h_T = \operatorname{diam} T$ satisfy the trace inequality

$$\|f\|_{L^2(E)} \lesssim h_T^{-1/2} \|f\|_{L^2(T)} + h_T^{1/2} \|\nabla f\|_{L^2(T)}.$$

Exercise 4

Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : |x| < 1, x_1, x_2 > 0\}$, $S = [0,1] \times \{0\}$ and

$$u_\epsilon : \Omega \to \mathbb{R}^2, x \mapsto \frac{[-x_2, x_1]^\top}{\varepsilon + |x|^2}.$$

a) Show that $u_\epsilon \in H(\text{div}, \Omega)$ and that there exists $\alpha_\epsilon \in \mathbb{R}$ such that $\widetilde{u}_\epsilon = \alpha_\epsilon u_\epsilon \in H(\text{div}, \Omega)$ and $\|\widetilde{u}_\epsilon\|_{L^2(\Omega)} = 1$.
b) Show that $\|\widetilde{u}_\epsilon \cdot n\|_{L^1(S)}$ is unbounded as $\epsilon \to 0$, and conclude that the trace operator for functions in $H(\text{div}, \Omega)$ is not well-defined as an operator into $L^1(\partial\Omega)$ in general.
c) Why is the expression $\int_{\partial\Omega} u_\epsilon \cdot n ds$ well-defined?