Exercise Sheet 1

Discussion on 24.10.22

Exercise 1 (Errors of difference quotients)
Let $J \in \mathbb{N}$, $\Delta x := 1/J$ and $x_j := j\Delta x$ for $j = 0,\dots,J$. Furthermore, let $u \in C^4([0,1])$. Show that for all $j \in \{0,\dots,J\}$ it holds

$$|\partial^{\pm} u(x_j) - u'(x_j)| \le \frac{\Delta x}{2}\|u''\|_{C([0,1])}$$
$$|\hat{\partial} u(x_j) - u'(x_j)| \le \frac{(\Delta x)^2}{6}\|u'''\|_{C([0,1])}$$
$$|\partial^{+}\partial^{-} u(x_j) - u''(x_j)| \le \frac{(\Delta x)^2}{12}\|u^{(4)}\|_{C([0,1])}.$$

Exercise 2 (Error estimate for implicit Euler scheme)
Additionally to the notation of Exercise 1.1, let $T > 0$, $K \in \mathbb{N}$, $\Delta t := T/K$ and $t_k := k\Delta t$ for $k = 0,\dots,K$. Prove that for $u \in C^4([0,T]\times[0,1])$ and $k = 0,\dots,K$, the $(U_j^k)_{jk}$ from the implicit Euler scheme satisfy

$$\sup_{j=0,\dots,J} |u(t_k, x_j) - U_j^k| \le \frac{t_k}{2}(\Delta t + (\Delta x)^2)\big(\|\partial_x^4 u\|_{C([0,T]\times[0,1])} + \|\partial_t^2 u\|_{C([0,T]\times[0,1])}\big).$$

Exercise 3 (Integration by parts)
a) Let $\Delta x > 0$, $(V_j)_{j=0,\dots,J} \in \mathbb{R}^{J+1}$, and $(W_j)_{j=0,\dots,J} \in \mathbb{R}^{J+1}$ with $V_0 = V_J = W_0 = W_J = 0$. Prove that

$$\sum_{j=1}^{J-1} \Delta x \left(\frac{V_{j+1} - 2V_j + V_{j-1}}{(\Delta x)^2}\right) W_j = -\sum_{j=0}^{J-1} \Delta x \left(\frac{V_{j+1} - V_j}{\Delta x}\right)\left(\frac{W_{j+1} - W_j}{\Delta x}\right).$$

b) Let $\Omega \subset \mathbb{R}^n$, $n = 1,2,3$ be a bounded domain with piecewise smooth boundary $\partial\Omega$, with outward normal ν along $\partial\Omega$. For $n = 1$, let $\operatorname{div} = \nabla$. For $v \in C^1(\overline{\Omega})$, $q \in C^1(\overline{\Omega};\mathbb{R}^n)$, show

$$\int_\Omega (v \operatorname{div} q + \nabla v \cdot q)\,\mathrm{d}x = \int_{\partial\Omega} v q \cdot \nu \,\mathrm{d}s.$$

Hint: You may assume that Gauss's divergence theorem holds for bounded domains with piecewise smooth boundary.

Exercise 4 (Discrete inverse inequality)
a) Let $\Delta x > 0$ and $(V_j)_{j=0,\dots,J} \in \mathbb{R}^{J+1}$ with $V_0 = V_J = 0$. Prove that

$$\sum_{j=0}^{J-1} \Delta x \left(\frac{V_{j+1} - V_j}{\Delta x}\right)^2 \le \frac{4}{(\Delta x)^2} \sum_{j=0}^{J} \Delta x\, V_j^2.$$

(b) Let $p \in P_k([a,b])$ be a polynomial of degree k on the interval $[a,b]$ with $a,b \in \mathbb{R}$ and $b > a$. Prove that for a constant $C > 0$ it holds

$$\|\partial_x p\|_{L^2([a,b])} \leq \frac{C}{b-a}\|p\|_{L^2([a,b])}.$$