Convergence rates in the local renewal theorem.

Artëm Sapozhnikov

Centrum voor Wiskunde en Informatica, Amsterdam

We improve results from [2] on subgeometric convergence rates in the local renewal theorem. The results are used in [3] to generalize previous results on convergence rates for Markov chains [4].

Key words and phrases. Renewal theorem; Convergence rates; Coupling.

1 Introduction

We study subgeometric convergence rates in the local renewal theorem, which plays an important role in obtaining results on convergence rates for Markov chains [4]. Let $\{\xi_n\}_{n\geq 0}$ and $\{\xi_n^{'}\}_{n\geq 0}$ be two sequences of mutually independent and integer-valued r.v.'s. Assume further that
(i) ξ_0 and $\xi_0^{'}$ are non-negative r.v.'s;
(ii) all $\{\xi_n\}_{n\geq 1}$ and $\{\xi_n^{'}\}_{n\geq 1}$ are i.i.d. and strictly positive, with a common distribution $p_k = \mathbf{P}(\xi_1 = k),\ k \geq 1$ with a finite mean $a \geq 1$ and such that

$$G.C.D.\{k \geq 1 \ :\ p_k > 0\} = 1.$$

Let

$$S_n = \sum_{k=0}^{n} \xi_k \quad \text{and} \quad S_n^{'} = \sum_{k=0}^{n} \xi_k^{'}, \quad n = 0, 1, \ldots$$

For any sample path, define a coupling time

$$T = \min\{l \geq 0 \ :\ l = S_n = S_m^{'} \ \text{ for some } \ n \ \text{ and } \ m\} \leq \infty.$$

Let $g \ :\ \mathbb{R}_+ \to \mathbb{R}_+$ be any nondecreasing function such that

$$g(x+y) \leq g(x)g(y) \ \text{ for all } x, y \in \mathbb{R}_+, \tag{1}$$

and

$$\frac{\ln g(x)}{x} \to 0, \quad \text{as } x \to \infty. \tag{2}$$

Put $g^0(x) = \int_0^x g(y)dy$. The following result indicates how fast two independent renewal processes started from different initial positions couple.

Theorem 1. *If* $\mathbf{E}g(\xi_0)$, $\mathbf{E}g({\xi'}_0)$, *and* $\mathbf{E}g^0(\xi_1)$ *are finite, then* $\mathbf{E}g(T)$ *is finite.*

[0]This research was supported by Science Foundation Ireland Grant No. SFI04/RP1/I512.
Address: Centrum voor Wiskunde en Informatica, 413 Kruislaan, 1098SJ Amsterdam, The Netherlands.
Email: Artem.Sapozhnikov@cwi.nl.

In case the ratio $\ln g(x)/x$ is monotone decreasing to 0 Theorem 1 is proved in [2]. Note that if $\ln g(x)/x$ is monotone decreasing to 0 then the following properties hold:

$$g(x+y) \le g(x)g(y) \quad \text{for all } x, y;$$

for any $\varepsilon > 0$ and $n > 0$ there exists $c = c(\varepsilon, n)$ such that for all x, y,

$$g(x+y) \le (1+\varepsilon)g(x) + g(x)g(y)\mathbf{I}_{\{y>n\}} + c(\varepsilon, n), \tag{3}$$

which is the main inequality used in the proof of convergence rates in the renewal theorem (see [2]).

Our proof of Theorem 1 is elementary and essentially based on the properties (1) and (2) of the function g, while the previous proofs [2, 4] use (3), which requires that $\ln g(x)/x$ is monotone decreasing to 0.

Remark 1. Theorem 1 holds for functions g_1 for which there is a function g satisfying conditions of Theorem 1, and $c \le g_1(x)/g(x) \le C$ for some positive c and C.

Remark 2. Condition (1) is essential. It assures that for two independent nonnegative random variables ξ and η such that $\mathbf{E}g(\xi) < \infty$ and $\mathbf{E}g(\eta) < \infty$ their sum satisfies $\mathbf{E}g(\xi+\eta) < \infty$.

Corollary 1. *Let $|\xi_0 - \xi_0'|$ be bounded a.s.. If $\mathbf{E}g^0(\xi_0)$, $\mathbf{E}g^0({\xi'}_0)$, and $\mathbf{E}g^0(\xi_1)$ are finite, then $\mathbf{E}g^0(T)$ is finite.*

We give the proofs in the next section.

2 Proofs.

The proof of Theorem 1 is based on two lemmas.

Lemma 1. *T is finite a.s. Moreover, for any $\gamma \in (0, 1/a)$, one can define two sequences $\{\psi_i\}_{i\ge1}$ and $\{V_i\}_{i\ge0}$ of i.i.d. non-negative integer-valued r.v.'s such that*
1) all r.v.'s ξ_0, $\xi_0^{'}$, $\{\psi_i\}_{i\ge1}$, and $\{V_i\}_{i\ge0}$ are mutually independent;
2) $\{\psi_i\}_{i\ge1}$ is an i.i.d. sequence with distribution

$$\mathbf{P}(\psi_i = 0) = \gamma, \quad \mathbf{P}(\psi_i \ge k) = \min\{1-\gamma, \sum_{j=k}^{\infty} \mathbf{P}(\xi_1 \ge j)\}, \quad k = 1, 2, \ldots;$$

3) there exists a number $n_0 = n_0(\gamma)$ such that

$$V_i \stackrel{\mathcal{D}}{=} \xi_1 + \ldots + \xi_{n_0};$$

4) for $\mu = \min\{n \ge 1 \ : \ \psi_n = 0\}$,

$$T \le_{st} \widehat{T} \equiv \max(\xi_0, \xi_0^{'}) + V_0 + \sum_{i=1}^{\mu-1}(\psi_i + V_i). \tag{4}$$

Proof of Lemma 1. Without loss of generality we can assume that $\xi_0 = 0$ a.s. Since $\{\xi_n\}_{n\geq 1}$ is aperiodic, there exists $n_0 \geq 0$ and $\gamma \in (0, 1/a)$ such that

$$\mathbf{P}(S_k = n \text{ for some } k) \geq \gamma > 0 \quad \text{for all } n \geq n_0.$$

We use the notation of [1, Theorem 4.2, Chapter 2]. Let $A_0 = 0$,

$$B_{2n} = \min\{S'_j - A_{2n} \ ; \ S'_j - A_{2n} \geq 0\} = S'_{\nu_{2n}} - A_{2n},$$

in particular, $B_0 = \xi'_0$,

$$V_{2n} = \xi'_{\nu_{2n}+1} + \ldots + \xi'_{\nu_{2n}+n_0},$$

$$A_{2n+1} = S'_{\nu_{2n}+n_0} = A_{2n} + B_{2n} + V_{2n},$$

$$B_{2n+1} = \min\{S_j - A_{2n+1} \ ; \ S_j - A_{2n+1} \geq 0\} = S_{\nu_{2n+1}} - A_{2n+1},$$

$$V_{2n+1} = \xi_{\nu_{2n+1}+1} + \ldots + \xi_{\nu_{2n+1}+n_0},$$

$$A_{2n+2} = S_{\nu_{2n+1}+n_0} = A_{2n+1} + B_{2n+1} + V_{2n+1}.$$

You can read more about the construction in [1, Theorem 4.2, Chapter 2].
Let $\tau = \min\{k \geq 0 \ : \ B_k = 0\}$. Then $\mathbf{P}(\tau \geq k) \leq (1-\gamma)^k$, and

$$T \leq A_\tau = \sum_{n=0}^{\tau-1} (B_n + V_n). \tag{5}$$

Note that, $\{V_n\}$ are i.i.d.
Let $D_n = \min\{k \geq 1 \ : \ S_k \geq n\}$ be the overshoot at the level n for $\{S_n\}$. Then

$$\mathbf{P}(D_n = k) \leq \mathbf{P}(\xi_1 \geq k) \quad \text{for all } n \geq 0.$$

Indeed,

$$\begin{aligned}
\mathbf{P}(D_n = k) &= \sum_{j=1}^{\infty} \mathbf{P}(S_{j-1} < n, \ S_j = n + k) = \sum_{j=1}^{\infty} \sum_{i=1}^{n-1} \mathbf{P}(S_{j-1} = i, \ \xi_j = n + k - i) \\
&= \sum_{i=1}^{n-1} \mathbf{P}(\xi_1 = n + k - i) \mathbf{P}(D_i = 0) \leq \sum_{i=1}^{n-1} \mathbf{P}(\xi_1 = k + i) \leq \mathbf{P}(\xi_1 \geq k).
\end{aligned}$$

Consider a sequence of i.i.d. random variables $\{\psi_n\}_{n\geq 1}$ independent of $\{V_n\}_{n\geq 0}$, where $\mathbf{P}(\psi_1 = 0) = \gamma$, and

$$\mathbf{P}(\psi_1 \geq k) = \min\Big(\sum_{i=k}^{\infty} \mathbf{P}(\xi_1 \geq i), \ 1 - \gamma\Big) \quad \text{for } k \geq 1.$$

Then

$$\mathbf{P}(D_n \geq k) \leq \mathbf{P}(\psi_1 \geq k) \quad \text{for all } n \geq n_0.$$

Note that

$$\begin{aligned}
\mathbf{P}(B_n \geq l \quad &| \quad B_0 = l_0, \ldots, B_{n-1} = l_{n-1}) = \mathbf{P}(B_n \geq l) \mid B_{n-1} = l_{n-1}) \\
&= \quad \mathbf{P}(D_{l_{n-1}+V_{n-1}} \geq l) \leq \mathbf{P}(\psi_n \geq l).
\end{aligned}$$

Hence for any $f(x_0, \dots, x_n)$, which is monotone function of each x_i, $0 \le i \le n$

$$\mathbf{E} f(B_0, \dots, B_n) \le \mathbf{E} f(\xi_0', \psi_1, \dots, \psi_n).$$

In particular,

$$\mathbf{P}(\sum_{i=0}^{\min(\tau,n)-1} (B_i + V_i) \ge k) \le \mathbf{P}(\xi_0' + V_0 + \sum_{i=1}^{\min(\mu,n)-1} (\psi_i + V_i) \ge k),$$

where $\mu = \min\{n \ge 1 \ : \ \psi_n = 0\}$, $\mathbf{P}(\mu \ge k) = (1-\gamma)^{k-1}$.
Hence

$$\mathbf{P}(\sum_{i=0}^{\tau-1} (B_i + V_i) \ge k) \le \mathbf{P}(\xi_0' + V_0 + \sum_{i=1}^{\mu-1} (\psi_i + V_i) \ge k).$$

The result immediately follows from (5). □

Lemma 2. *Let $g \ : \ \mathbb{R} \to \mathbb{R}_+$ be any nondecreasing function such that $g(x) = 1$ for $x \le 0$, $g(x+y) \le g(x)g(y)$ for all $x, y \in \mathbb{R}_+$ and $\ln g(x)/x \to 0$ as $x \to \infty$.*
Let $\{\xi_n\}$ be a sequence of i.i.d. random variables on $\mathbb{R}_+$ such that $\mathbf{E}g(\xi_1) < \infty$, μ be a random variable which is independent of ξ_n and $\mathbf{P}(\mu = k) = p(1-p)^{k-1}$ for some $p \in (0,1)$. Then

$$\mathbf{E} g(\sum_{i=1}^{\mu} \xi_i) < \infty.$$

Proof of Lemma 2. We consider

$$\mathbf{E} g\left(\sum_{i=1}^{\mu} \xi_i \right) = \frac{p}{1-p} \sum_{n=1}^{\infty} \mathbf{E} g\left(\sum_{i=1}^{n} \xi_i \right) (1-p)^n.$$

Let $X_n = \sum\limits_{i=1}^{n} \xi_i$, $q = \ln \frac{1}{1-p} > 0$, and $G(x) = \ln g(x)$. Then

$$\mathbf{E} g\left(\sum_{i=1}^{n} \xi_i \right) (1-p)^n = \mathbf{E} e^{G(X_n) - qn} = \mathbf{E} e^{G(X_n) - \frac{2}{3}qn} e^{-\frac{1}{3}qn}.$$

Hence it is sufficient to show that

$$\sup_n \mathbf{E} e^{G(X_n) - \frac{2}{3}qn} < \infty.$$

Note that there exists $K > 0$ such that

$$\mathbf{E} e^{G(\xi_1 - K)} \le e^{\frac{1}{3}q}.$$

Since $G(x) = o(x)$ as $x \to \infty$, there exists $\tilde{x} > 0$ such that $G(Kx) \le qx/3$ for all $x \ge \tilde{x}$. Take $n \ge \tilde{x}$. Then

$$\begin{aligned} \mathbf{E} e^{G(X_n) - \frac{2}{3}qn} &\le \mathbf{E} e^{G(X_n) - G(Kn) - \frac{1}{3}qn} \le \mathbf{E} e^{G(X_n - Kn) - \frac{1}{3}qn} \\ &\le \mathbf{E} e^{\sum\limits_{i=1}^{n} G(\xi_i - K) - \frac{1}{3}qn} = \left(\mathbf{E} e^{G(\xi_1 - K)} \right)^n e^{-\frac{1}{3}qn} \le 1. \end{aligned}$$

Finally, note that $\mathbf{E}g\left(\sum_{i=1}^{n}\xi_i\right) \le (\mathbf{E}g(\xi_1))^n < \infty$ for all n, which completes the proof. $\square$

We have enough tools now to prove the main theorem.

Proof of Theorem 1. Consider i.i.d. random variables $\{\zeta_n\}_{n\ge1}$ independent of ξ_0, ξ_0', $\{\psi_n\}_{n\ge1}$, and $\{V_n\}_{n\ge0}$ such that

$$\mathbf{P}(\zeta_i = k) = \mathbf{P}(\psi_i + V_i = k \mid \psi_i \ne 0), \quad i \ge 1.$$

Then

$$\mathbf{P}(\sum_{i=1}^{\mu-1}\zeta_i = k) = \mathbf{P}(\sum_{i=1}^{\mu-1}(\psi_i + V_i) = k).$$

Hence

$$T \le_{st} \max(\xi_0, \xi_0') + V_0 + \sum_{i=1}^{\mu-1}\zeta_i.$$

Since $\mathbf{P}(\zeta_1 = k) \le \frac{1}{1-\gamma}\mathbf{P}(\psi_1 + V_1 = k)$ and $g(n+m) \le g(n)g(m)$ for all $n, m \in \mathbb{Z}_+$, it follows that $\mathbf{E}g(\zeta_1) < \infty$. Then (see Lemma 2)

$$\mathbf{E}g(\sum_{i=1}^{\mu-1}\zeta_i) < \infty.$$

Hence $\mathbf{E}g(T) < \infty$. $\square$

Proof of Corollary 1. Let ξ_0 and ξ_0' be independent and distributed according to the stationary renewal process. Since $\mathbf{E}g^0(\xi_1) < \infty$, it follows from Theorem 1 that $\mathbf{E}g(T) < \infty$. Consider a new renewal process η_n defined as follows. We say that renewal takes place at point n, if so for both original processes. The new process is a stationary renewal process with delay $\eta_0 = T$ and increments η_n distributed according to the first coupling time of two independent undelayed renewal processes. It is well-known that $\mathbf{E}g(\eta_0) < \infty$ if and only if $\mathbf{E}g^0(\eta_1) < \infty$. Hence the corollary is proved in the particular case of undelayed renewal processes. Let $T_{x,y}$ be the coupling time for two independent renewal processes with delays x and y. Then it is clear that

$$\mathbf{E}g^0(T_{0,0}) \ge \mathbf{E}g^0(T_{n_1,n_1'})\mathbf{P}(S_k = n_1,\ S'_{k'} = n_1' \text{ for some } k, k';\ T_{0,0} \ge \max(n_1, n_1')),$$

where $\mathbf{P}(S_k = n_1,\ S'_{k'} = n_1' \text{ for some } k, k';\ T_{0,0} \ge \max(n_1, n_1'))$ is positive for all large enough n_1 and n_1' (from the proof of Lemma 1 it follows that it is sufficient to take $n_1, n_1' \ge n_0$). hence the corollary is true in case the delays are $\xi_0 = i$ and $\xi_0' = j$ for any i and j. The rest of the proof is now obvious. $\square$

References

[1] Lindvall, T. (1992), *Lectures on the coupling method.* (Wiley, New York).

[2] Lindvall, T. (1979), On coupling of discrete renewal processes. *Z. Wahrscheinlichkeitsth.* **48**, 57–70.

[3] Sapozhnikov, A. (2006), Subgeometric rates of convergence of f-ergodic Markov chains. *Submitted.*

[4] Tuominen, P. and Tweedie, R.L. (1994) Subgeometric rates of convergence of f-ergodic Markov chains. *Adv.Appl.Prob* **26**, 775–798.