On the truncated long range percolation on $\mathbb{Z}^2$

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Abstract

We consider an independent long range bond percolation on $\mathbb{Z}^2$. Horizontal and vertical bonds of length n are independently open with probability $p_n \in [0,1]$. Given $\sum_{n=1}^{\infty}\prod_{i=1}^{n}(1-p_i) < \infty$, we prove that there exists an infinite cluster of open bonds of length $\le N$ for some large but finite N. The result gives a partial answer to the conjecture from [2].

1 Notation and Results

We consider an independent bond percolation on the graph $\mathcal{G} = (\mathbb{Z}^2, \mathcal{E})$, where $\mathcal{E} = \{< x, y > \in \mathbb{Z}^2 \times \mathbb{Z}^2 \ : \ x \neq y$ and $x_1 = y_1$ or $x_2 = y_2\}$. For a given sequence (p_n) such that $p_n \in [0,1]$, we declare an edge $< x, x+ne_i >$ ($x \in \mathbb{Z}^2$, $i \in \{1,2\}$) to be open with probability p_n and closed otherwise. More formally, we consider the probability space $(\Omega, \mathcal{F}, P)$. As sample space we take $\Omega = \{0,1\}^{\mathcal{E}}$. It's elements are denoted as $\omega = \{\omega(f) \ : \ f \in \mathcal{E}\}$. The value $\omega(f) = 1$ corresponds to f being open, and the value $\omega(f) = 0$ corresponds to f being closed. We take $\mathcal{F}$ to be the σ-algebra generated by finite cylinder sets in Ω. We define the product measure P on $(\Omega, \mathcal{F})$ as $\prod_{f\in\mathcal{E}} \mu_f$, where μ_f is Bernoulli measure on $\{0,1\}$ given by

$$\mu_f(\omega(f) = 1) = 1 - \mu_f(\omega(f) = 0) = p_{|f|},$$

where $|f| = \max(|x_1 - y_1|, |x_2 - y_2|)$ given $f =< x, y >$.

Definition 1.1. We say that two sites $x, y \in \mathbb{Z}^2$ are *k-connected*, $x \overset{k}{\longleftrightarrow} y$, if there are $v_1, \ldots, v_m \in \mathbb{Z}^2$ such that $v_1 = x$, $v_m = y$, $< v_i, v_{i+1} > \in \mathcal{E}$ is open, and $|v_i - v_{i+1}| \le k$ for all i. If $k = \infty$, then we say that x and y are *connected*, $x \longleftrightarrow y$. We say that two sites x and y of $\mathbb{Z}^2$ are *connected in* $W \subset \mathbb{Z}^2$ if $x, y \in W$, and there is an open path between x and y such that all the sites of the path are in W.

In this note we study the well known truncation problem: given a sequence (p_n) for which $\mathbf{P}(0 \leftrightarrow \infty) > 0$, is it true that $\mathbf{P}(0 \overset{N}{\leftrightarrow} \infty) > 0$ for some large finite N? The answer is no for the one-dimensional independent percolation ([4, 10]). It is believed that the answer is yes in dimensions $d \ge 2$. However, only partial results have been obtained so far.

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In [8] the affirmative answer is given in the case the sequence (p_n) is exponentially decaying. The heavy-tailed case has been considered in [1, 9, 11]. In all the papers it is assumed that the sequence (p_n) is monotone decreasing with some conditions on the speed of decay. The first results for non-monotone sequences were obtained in [2, 3]. In [3] the positive answer to the truncation question was given for sequences (p_n) such that $\limsup_n p_n > 0$. For non-summable sequences (p_n) ($\sum_n p_n = \infty$), the affirmative answer to the truncation question was given in [2] in dimensions $d \geq 3$. It was also conjectured that the statement is true in two dimensions. In this note we answer yes to the truncation question in two dimensions for a very general class of non-summable sequences (p_n) (see e.g. condition (1.3)), which supports the conjecture from [2]. Our approach is different from the one in [3]. It is based on Blackwell's renewal theorem and renormalization techniques.

Theorem 1.1. *Given a sequence (p_n) such that $p_n \in [0,1]$ and $\sum_{n=1}^{\infty} p_n = \infty$, if*

$$\limsup_{n\to\infty} \mathbf{P}(0 \text{ and } n \text{ are connected in } [0,n]) > 0 \tag{1.1}$$

then there exists N such that

$$\mathbf{P}(0 \overset{N}{\longleftrightarrow} \infty) > 0. \tag{1.2}$$

Remark 1. If $\limsup_n p_n > 0$ then (1.1) is trivially satisfied. In particular, the result from [2] follows.

In the next theorem we give a sufficient condition for (1.1).

Theorem 1.2. *If*

$$\sum_{n=1}^{\infty} \prod_{i=1}^{n} (1-p_i) < \infty \tag{1.3}$$

then condition (1.1) holds.

2 Proofs.

Proof of Theorem 1.1. We assume for convenience that the greatest common divisor G.C.D. $\{k \ : \ p_k > 0\} = 1$. The condition ensures that the infinite open cluster is unique (see [5, Theorem 12.3]). If G.C.D. $\{k \ : \ p_k > 0\} = m > 1$ then we consider the bond percolation on $m\mathbb{Z}^2$ with

$$\mathbf{P}(< mx, m(x+ne_1) > \text{ is open }) = \mathbf{P}(< mx, m(x+ne_2) > \text{ is open }) = p_{mn}.$$

For the sake of notation, we also assume that there exist $p > 0$ and n_0 such that, for all $n \geq n_0$,

$$\mathbf{P}(0 \text{ and } n \text{ are connected in } [0,n]) \geq p. \tag{2.1}$$

The general case when (2.1) is only satisfied for an infinite subsequence (n_k) can be treated in the same way.

The proof is based on a renormalization argument. Let l and L be positive integers such that $l < L$. For any $x \in \mathbb{Z}^2$, the event A_x occurs if:

- any two sites from the set $2Lx + [-l,l] \times \{0\}$ are connected in $2Lx + [-L,L] \times \{0\}$ (see Definition 1.1); and
- any two sites from the set $2Lx + \{0\} \times [-l,l]$ are connected in $2Lx + \{0\} \times [-L,L]$.

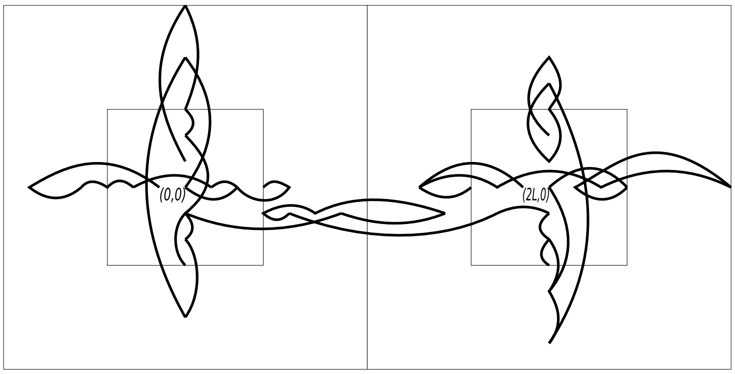

Figure 1: Events A_0, A_{e_1} and B_{0,e_1}. The inner boxes are $B(0,l)$ and $B(2Le_1,l)$. The outer boxes are $B(0,L)$ and $B(2Le_1,L)$.

The events A_0 and A_{e_1} are illustrated in Figure 1. From the space homogeneity it follows that $\mathbf{P}(A_x) = \mathbf{P}(A_0)$ for all $x \in \mathbb{Z}^2$. Moreover, since

$$\mathbf{P}(\text{all the sites } [-l,l] \text{ are connected in } \mathbb{Z}) = 1$$

for all $l \in \mathbb{N}$ (we use the assumption $G.C.D.\{i \,:\, p_i > 0\} = 1$), for any $\varepsilon > 0$ and for all $l \in \mathbb{N}$ there exists $L_1 = L_1(\varepsilon, l)$ such that, for all $L \geq L_1$,

$$\mathbf{P}(A_0) > 1 - \varepsilon.$$

For $x \in \mathbb{Z}^2$ and $y = x + (1,0)$, we say that the event $B_{x,y}$ occurs if:

- there exists $k \in [-l,l]\backslash\{0\}$ such that the sites $2Lx + (0,k)$ and $2Ly + (0,k)$ are connected in $[2Lx + (0,k), 2Ly + (0,k)] = 2Lx + (0,k) + [0,2L] \times \{0\}$.

In Figure 1, the event B_{0,e_1} occurs with $k = -1$. We assume that $B_{y,x} = B_{x,y}$. Similarly, for $x \in \mathbb{Z}^2$ and $y = x + (0,1)$, the events $B_{x,y}$ and $B_{y,x}$ occur if:

- there exists $k \in [-l,l]\backslash\{0\}$ such that the sites $2Lx + (k,0)$ and $2Ly + (k,0)$ are connected in $[2Lx + (k,0), 2Ly + (k,0)] = 2Lx + (k,0) + \{0\} \times [0,2L]$.

Space homogeneity and symmetry of the model imply that, for any $x \sim y$ (i.e. x and y are nearest neighbours in $\mathbb{Z}^2$) and $u \sim v$, $\mathbf{P}(B_{x,y}) = \mathbf{P}(B_{u,v})$.

Condition (2.1) implies that for all $L \geq n_0$,

$$\mathbf{P}(0 \text{ and } 2L \text{ are connected in } [0,2L]) \geq p > 0. \tag{2.2}$$

Therefore, for any $\varepsilon > 0$ there exist $l_0 = l_0(\varepsilon) \in \mathbb{N}$ and $L > \max(l_0, n_0)$ such that

$$\mathbf{P}(B_{0,e_1}) > 1 - \varepsilon.$$

For $\varepsilon > 0$, we take $L \geq \max(L_1(\varepsilon, l_0(\varepsilon)), n_0)$. It follows that

$$\mathbf{P}(A_0) > 1 - \varepsilon \;\text{ and }\; \mathbf{P}(B_{0,e_1}) > 1 - \varepsilon.$$

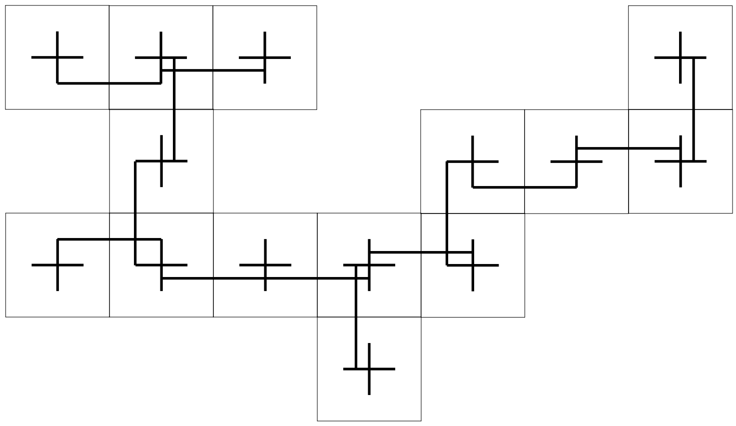

Figure 2: An example of renormalized cluster. Crosses correspond to occurence of events A_x, and connections between crosses correspond to occurence of events $B_{x,y}$.

Moreover, the events $\{A_x \ : \ x \in \mathbb{Z}^2\} \cup \{B_{y,z} \ : \ y, z \in \mathbb{Z}^2, y \sim z\}$ are defined in terms of the states of edges in disjoint subsets of $\mathcal{E}$ and therefore independent.

Now it is easy to finish the proof. If A_x occurs, we say that the site x is *open*. If $B_{y,z}$ occurs, we say that the bond $< y, z >$ is *open*. The constructed model is an independent nearest-neighbour site-bond percolation.

We can choose $\varepsilon > 0$ small enough such that there is an infinite open cluster in the renormalized site-bond percolation model (see e.g. [6]). The existence of an infinite open cluster in the renormalized model implies the existence of infinite open cluster of $2L$-connected sites in the original model. Therefore we can take $N = 2L$. □

Proof of Theorem 1.2. As in the proof of Theorem 1.1 we can assume without loss of generality that G.C.D. $\{k \ : \ p_k > 0\} = 1$. For any $x \in \mathbb{Z}^2$ we define

$$\xi_x = \min\{n \ : \ < x, x + ne_1 > \ \text{ is open } \}.$$

Note that ξ_x are i.i.d. random variables with distribution

$$\mathbf{P}(\xi_0 > n) = \prod_{i=1}^{n} (1 - p_i).$$

Since $\sum_n p_n = \infty$, the random variables are finite almost surely. Moreover,

$$\mathbf{E}\xi_x = \sum_{n=0}^{\infty} \prod_{i=1}^{n} (1 - p_i) < \infty.$$

An important result in renewal theory is Blackwell's theorem (see e.g. [7]):

Theorem 2.1. *Let $\{X_i\}$ be a sequence of i.i.d. random variables taking values in $\mathbb{Z}_+$, and $S_k = \sum_{i=1}^{k} X_i$. If*

$$\text{G.C.D. } \{k \ : \ \mathbf{P}(X_1 = k) > 0\} = 1$$

then

$$\mathbf{P}(\text{ there exists } k \text{ such that } S_k = n) \to \frac{1}{\mathbf{E}X_1},$$

as $n \to \infty$. Here, if $\mathbf{E}X_1 = \infty$ then the limit is 0.

From Blackwell's theorem we conclude that there exists n_0 such that for all $n \geq n_0$

$$\mathbf{P}\left(0 \text{ and } n \text{ are connected in } [0,n]\right) \geq \frac{1}{2\mathbf{E}\xi_0} = \left(2\sum_{n=0}^{\infty}\prod_{i=1}^{n}(1-p_i)\right)^{-1} > 0.$$

□

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References

[1] BERGER N. (2002). Transience, recurrence and critical behavior for long-range percolation. *Comm. Math. Phys.* **226**, 531–558.

[2] FRIEDLI S. AND DE LIMA B.N.B. (2006). On the truncation of systems with non-summable interactions. *J. Stat. Phys.* **122 (6)**, 1215–1236.

[3] FRIEDLI S., DE LIMA B.N.B. AND SIDORAVICIUS V. (2004). On long range percolation with heavy tails. *Elect. Comm. Prob.* **9**, 175–177.

[4] FRÖHLICH J. AND SPENCER T. (1982). The phase transition in the one-dimensional Ising model with $1/r^2$ interaction energy. *Comm. Math. Phys.* **84 (1)**, 87–101.

[5] GRIMMETT G.R. (1999). *Percolation*. 2nd edition, Springer, New York.

[6] HAMMERSLEY J.M. (1980). A generalization of McDiarmid's theorem for mixed Bernoulli percolation. *Math. Proc. Camb. Phil. Soc.* **88 (1)**, 167–170.

[7] LINDVALL T. (1992). *Lectures on the coupling method.* Wiley, New York.

[8] MEESTER R. AND STEIF J.E. (1996). On the continuity of the critical value for long range per- colation in the exponential case. *Comm. Math. Phys.* **180**, 483–504.

[9] MENSHIKOV M., SIDORAVICIUS V. AND VACHKOVSKAIA M. (2001). A note on two-dimensional truncated long-range percolation. *Adv. Appl. Prob.* **33**, 912-929.

[10] NEWMAN C.M. AND SCHULMAN L.S. (1986). One dimensional $1/|i-j|^s$ percolation models: the existence of a transition for $s \leq 2$. *Comm. Math. Phys.* **104**, 547–571.

[11] SIDORAVICIUS V., SURGAILIS D. AND VARES M.E. (1999). On the truncated anisotropic long-range percolation on $\mathbb{Z}^2$. *Stoch. Proc. and Appl.* **81**, 337–349.