Random functions on the hypercube: functional inequalities, noise sensitivity and sharp thresholds

Reading seminar

Fridays, 15:15–17:00, SG 2-14 (First meeting: 20.10.2017)

The aim of this seminar is to discuss classical results and recent developments about random functions on the hypercube. They arise naturally in theoretical computer science and combinatorics, and in the last decade their general properties have been instrumental for new striking developments in statistical physics and percolation.

Consider a Boolean function $f : \{0,1\}^n \to \{0,1\}$ and a Bernoulli-p product measure $\mathbb{P}_p$ on $\{0,1\}^n$. We will particularly discuss various tools used to understand two phenomena observed in some general classes of functions:

- *noise sensitivity* — after resampling every coordinate independently with probability ε, the (random) value of f becomes almost independent from the original one, namely, if ω is the random point on the hypercube sampled from $\mathbb{P}_p$, and ω_ε is its perturbation, then $\mathbb{E}[f(\omega)f(\omega_\varepsilon)] - \mathbb{E}[f(\omega)]^2 \to 0$ as $n \to \infty$ (the opposite notion to it is the *noise stability*),
- *sharp threshold* — for a monotone function f, as p increases from 0 to 1, the probability $\mathbb{P}_p[f(\omega) = 1]$ increases from almost 0 to almost 1 in an interval of length of order $\frac{1}{\log n}$.

A crucial role is played by the *influence* $\mathrm{I}_i^p(f)$ of individual bit i on the function—the probability that the value of the function changes after flipping the ith bit. A type of discrete Poincaré inequality easily gives $\mathrm{Var}(f) \le p(1-p)\sum_{i=1}^n I_i^p(f)$. This is too weak to imply any useful conclusions. In the first part of the seminar we will discuss how to obtain non-trivial improvements of this inequality using harmonic analysis on the hypercube, hypercontractivity, and randomized algorithms [3, 12, 14, 17]. (One of the implications of this theory states: when all the influences are small, then their sum is large.) In the second part we will use these results to link properties of influences to the above stated phenomena. If time permits and based on interests of participants, we may discuss applications in computer science and statistical physics, extensions to non-product measures, lower bounds on the variance (reverse Poincaré inequality), etc.

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