RETAKE SOLUTIONS, 05.10.2015

1. Give an example of a linear operator T on $\mathbb{R}^2$ such that $T^2 = T$ and $\mathrm{rank}(T) = 1$.

   Answer: Projection on any line through $(0, 0)$.

   Solution. Perhaps the simplest example is the projection operator T that maps each $(x_1, x_2) \in \mathbb{R}^2$ to $(x_1, 0)$.

   For any $(x_1, x_2) \in \mathbb{R}^2$, $T^2(x_1, x_2) = T(T(x_1, x_2)) = T(x_1, 0) = (x_1, 0) = T(x_1, x_2)$. Thus, $T^2 = T$.

   By definition, $\mathrm{rank}(T)$ is the dimension of the image of T, which is the maximal number of linearly independent vectors in $\mathrm{Im}(T)$. Since any two vectors $(x_1, 0)$ and $(y_1, 0)$ are linearly dependent, the dimension equals to 1. $\square$

   Remark. More generally, let $\langle \cdot, \cdot \rangle$ be the canonical inner product on $\mathbb{R}^2$, i.e., for any $u = (u_1, u_2)$ and $v = (v_1, v_2)$, $\langle u, v \rangle = u_1 v_1 + u_2 v_2$. Let $a = (a_1, a_2)$ be a unit vector in $\mathbb{R}^2$, i.e., $\langle a, a \rangle = 1$. Let T_a be the projection on the line spanned by a, i.e., for any $x = (x_1, x_2)$, $T_a(x) = \langle x, a \rangle a$. Then T_a is linear and satisfies the desired properties.

2. Consider the space $\mathbb{R}^2$ with the inner product $\langle u, v \rangle = u_1 v_1 + u_2 v_2$, where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are arbitrary elements of $\mathbb{R}^2$. Let T be the linear operator on $\mathbb{R}^2$ which maps each vector (v_1, v_2) to vector $(v_1 + v_2, v_1 - v_2)$.

   (a) Is T normal?

   (b) Find an orthogonal basis of $\mathbb{R}^2$ in which the matrix of T is diagonal.

   Answer: (a) Yes. (b) For example, $\left\{ \left(1, -(1 + \sqrt{2})\right), \left(1, -(1 - \sqrt{2})\right) \right\}$.

   Solution. (a) The matrix of T in the orthonormal basis $\{(1, 0), (0, 1)\}$ is $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Since the matrix is symmetric, the operator T is self-adjoint, and, in particular, normal.

   (b) Since T is normal, one can find in $\mathbb{R}^2$ an orthogonal basis $\{u, v\}$ of eigenvectors of T. Let λ and μ be the eigenvalues of T corresponding to u and v, respectively, i.e., $Tu = \lambda u$ and $Tv = \mu v$. Then the matrix of T in the basis $\{u, v\}$ is $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. Thus, the question is reduced to finding u and v.

   We first find the eigenvalues of T, namely, the roots of the characteristic polynomial $P_T(x) = \det \begin{pmatrix} 1 - x & 1 \\ 1 & -1 - x \end{pmatrix} = x^2 - 2$. The eigenvalues are $\lambda = -\sqrt{2}$ and

$\mu = \sqrt{2}$. To find u and v, we solve the systems of linear equations $(T - \lambda I)u = 0$ and $(T - \mu I)v = 0$:

$$\begin{pmatrix} 1+\sqrt{2} & 1 \\ 1 & -1+\sqrt{2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0, \qquad \begin{pmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.$$

We obtain $u_2 = -(1+\sqrt{2})u_1$ and $v_2 = -(1-\sqrt{2})v_1$. In particular, one can take

$$u = (u_1, u_2) = \left(1, -(1+\sqrt{2})\right), \qquad v = (v_1, v_2) = \left(1, -(1-\sqrt{2})\right).$$

□

3. Give an example of a bilinear form on the space of complex valued continuous functions on $[0, 1]$.

   Answer: Examples include $B(f, g) = \int_0^1 f(x)g(x)dx$, $B(f, g) = f(0)g(0)$.

4. For which of the following functions does the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ exist?

   (a) $f(x, y) = \frac{x^2}{x^2+y^2}$, for (x, y) with $x^2 + y^2 > 0$.

   (b) $f(x, y) = \frac{x^3}{x^2+y^2}$, for (x, y) with $x^2 + y^2 > 0$.

   Answer: (a) No. (b) Yes.

   Solution. (a) Let $f(x, y) = \frac{x^2}{x^2+y^2}$, for (x, y) with $x^2 + y^2 > 0$. If there exists $L = \lim_{(x,y)\to(0,0)} f(x, y)$, then for any choice of $\alpha = \alpha(t)$ and $\beta = \beta(t)$ with $\lim_{t\to 0}(\alpha(t), \beta(t)) = (0, 0)$, the limit $\lim_{t\to 0} f(\alpha(t), \beta(t))$ is the same and equals L.

   For $\alpha_1(t) = t$, $\beta_1(t) = 0$, $\lim_{t\to 0} f(\alpha_1(t), \beta_1(t)) = \lim_{t\to 0} \frac{t^2}{t^2} = 1$.

   For $\alpha_2(t) = t$, $\beta_2(t) = t$, $\lim_{t\to 0} f(\alpha_2(t), \beta_2(t)) = \lim_{t\to 0} \frac{t^2}{2t^2} = \frac{1}{2}$.

   Since the two limits are different, the limit of $f(x, y)$ at $(0, 0)$ does not exist.

   (b) We will prove that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$. For this it suffices to prove that $\lim_{(x,y)\to(0,0)} |f(x, y)| = 0$. Note that

   $$0 \leq |f(x, y)| = \frac{|x|^3}{x^2 + y^2} \leq \frac{|x|^3}{x^2} = |x|.$$

   Since $\lim_{(x,y)\to(0,0)} |x| = 0$, the result follows. □

5. Write the equation of the tangent plane to elliptic paraboloid $z = 2x^2 + y^2$ at the point $(0, 1, 1)$.

   Answer: $z = 2y - 1$.

Solution. The equation of the tangent plane to $z = f(x, y)$ at point (x_0, y_0, z_0) is

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

In our case, $f_x = 4x$, $f_y = 2y$. In particular, $f_x(0, 1) = 0$, $f_y(0, 1) = 2$, and the equation is $z = 1 + 0(x - 0) + 2(y - 1) = 2y - 1$. $\square$

6. Find the maximum and minimum of the function $z = xy$ on the circle $x^2 + y^2 = 1$.

   Answer: $z = \frac{1}{2}$ *and* $z = -\frac{1}{2}$.

   Solution. First note that the circle $x^2 + y^2 = 1$ is a closed and bounded set in $\mathbb{R}^2$, thus z attains its maximum and minimum at some points of the circle.

   To find the maximum and minimum of z, we use the method of Lagrange multipliers. For $\lambda \in \mathbb{R}$, let $F(x, y) = xy - \lambda(x^2 + y^2 - 1)$. If z attains its maximum or minimum at (x_*, y_*) on the circle $x^2 + y^2 = 1$, then $F_x(x_*, y_*) = F_y(x_*, y_*) = 0$. Thus, the points where z attains its maximum and minimum, are solutions to the following system of equations for some $\lambda \in \mathbb{R}$:

$$\begin{cases} y - \lambda 2x = 0, \\ x - \lambda 2y = 0, \\ x^2 + y^2 = 1. \end{cases}$$

   From the first two equations, $y = 4\lambda^2 y$ and $x = 4\lambda^2 x$. Since either $x \neq 0$ or $y \neq 0$, we must have $4\lambda^2 = 1$ and $\lambda = \pm\frac{1}{2}$. Thus, either $x = y = \pm\frac{1}{\sqrt{2}}$ or $x = -y = \pm\frac{1}{\sqrt{2}}$. In the first case, $z = \frac{1}{2}$, in the second, $z = -\frac{1}{2}$. $\square$

7. Does the following series converge uniformly on $\mathbb{R}$?

$$\sum_{n=1}^{\infty} e^{-n^3|x|} \sin nx.$$

   Answer: *Yes.*

   Solution. Let $u_n(x) = e^{-n^3|x|} \sin nx$. We will prove that for all $x \in \mathbb{R}$, $|u_n(x)| \leq \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, the Weierstrass criterion for uniform convergence will imply that the given series converges uniformly on $\mathbb{R}$.

   We first recall that $|\sin nx| \leq |nx|$ for all x. Thus, it suffices to prove that $v_n(x) = e^{-n^3|x|}|x| \leq \frac{1}{n^3}$. Since $v_n(-x) = v_n(x)$, it suffices to consider only $x > 0$.

   We find the supremum of $v_n(x)$ for $x > 0$. We compute $v_n'(x) = (e^{-n^3 x} x)' = -n^3 e^{-n^3 x} x + e^{-n^3 x}$, which equals to 0 at $x = \frac{1}{n^3}$. Since $v_n(x)$ is non-negative for $x > 0$ and takes values arbitrarily close to 0 (near 0 and infinity), $x = \frac{1}{n^3}$ is the point of maximum of $v_n(x)$. In particular, $v_n(x) \leq v_n(\frac{1}{n^3}) = e^{-1}\frac{1}{n^3} \leq \frac{1}{n^3}$. This is what remained to prove. $\square$

8. Let (X, ρ) be a metric space. Is $\sqrt{\rho}$ a metric on X?

   Answer: Yes.

   Solution. For $x, y \in X$, let $g(x, y) = \sqrt{\rho(x, y)}$. Then g is a metric on X if (a) $g(x, y) \geq 0$ for all $x, y \in X$ and $g(x, y) = 0$ iff $x = y$, (b) $g(x, y) = g(y, x)$ for all $x, y \in X$, (c) $g(x, y) \leq g(x, z) + g(z, y)$ for all $x, y, z \in X$.

   The properties (a) and (b) are immediate, since they are satisfied by ρ. To prove (c), let $x, y, z \in X$. Since ρ is a metric, we have

   $$g(x, y)^2 = \rho(x, y) \leq \rho(x, z) + \rho(z, y) = g(x, z)^2 + g(z, y)^2 \leq (g(x, z) + g(z, y))^2.$$

   This proves that g satisfies (c). Thus, g is a metric on X. □

9. Find the general solution to the first order differential equation $y' - y = 2x + 1$.

   Answer: $y = Ce^x - 2x - 3$.

   Solution. This is an equation of the form $y' = f(ax + by)$, thus we make a substitution $z = 2x + y$. Then, $dz = 2dx + dy$, and the equation becomes $dz - 2dx = (z+1)dx$, or, equivalently, $dz = (z + 3)dx$. Separation of variables and integration gives $\ln|z + 3| = x + \ln|C|$ or $z + 3 = Ce^x$, which in the original variables gives the solution $y = Ce^x - 2x - 3$, for any $C \neq 0$.

   Note that in the above calcuation we divided by $z + 3$. A direct check gives that $z = -3$ is also a solution, but it corresponds to $C = 0$ in the above general formula. Thus, the general solution is $y = Ce^x - 2x - 3$, for any C. □

   Remark. Another way to solve this equation is to notice that the equation is linear, thus its solution is the sum of its particular solution $(-2x - 3)$ and the general solution to the homogeneous equation $y' - y = 0$ (Ce^x).

10. Find the general solution to the second order differential equation $y'' + 2y' + y = x$.

    Answer: $y = C_1e^{-x} + C_2xe^{-x} + x - 2$.

    Solution. This is a linear equation, thus its general solution equals to the sum of its particular solution and the general solution to the homogeneous equation $y'' + 2y' + y = 0$.

    Consider first the homogeneous equation $y'' + 2y' + y = 0$. The corresponding characteristic polynomial is $\lambda^2 + 2\lambda + 1$. Its root is $\lambda = -1$ of multiplicity 2. Thus, the general solution to the homogeneous equation is $y_1(x) = C_1e^{-x} + C_2xe^{-x}$.

    To find a particular solution to the inhomogeneous equation, note that the right hand side has the form $(ax + b)e^{0x}$. Since 0 is not a root of the characteristic polynomial, we look for a particular solution in the form $y_2(x) = ax + b$. Substitution in the equation gives

    $$(ax + b)'' + 2(ax + b)' + (ax + b) = x,$$

    from which $a = 1$ and $b = -2$. Thus, $y_2(x) = x - 2$.

    Finally, the general solution $y(x) = y_1(x) + y_2(x) = C_1e^{-x} + C_2xe^{-x} + x - 2$. □