EXERCISES 9.1-9.2 (submit by 12.06.2015)

1. Let $f_n(x) \rightrightarrows f(x)$ and $g_n(x) \rightrightarrows g(x)$ on E. Prove that $(f_n + g_n)(x) \rightrightarrows (f+g)(x)$ and $(f_n g_n)(x) \rightrightarrows (fg)(x)$ on E.

2. Prove that if $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on E then $u_n(x) \rightrightarrows 0$ on E.

3. In each of the cases below, find the limit $f(x)$ of $f_n(x)$ for $x \in [0,1]$. Does f_n converge to f uniformly on $[0,1]$? Is $f(x)$ continuous? Does $\int_0^1 f_n(x)dx$ converge to $\int_0^1 f(x)dx$? Justify your answers.

   (a) $f_n(x) = \frac{1}{1+nx}$
   (b) $f_n(x) = \frac{x}{1+n^2x^2}$
   (c) $f_n(x) = \frac{nx}{1+n^2x^2}$
   (d) $f_n(x) = n^2 x e^{-n^2x^2}$
   (e) $f_n(x) = x e^{-n^2x^2}$.

4. Which of the following series converge uniformly on $\mathbb{R}$? Which of the sums are continuous functions of x on $\mathbb{R}$?

   (a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^2}{(1+x^2)^n}$
   (b) $\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$
   (c) $\sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$
   (d) $\sum_{n=1}^{\infty} \frac{\sin nx}{x^2+n}$.

5. For which of the following series $\sum_{n=1}^{\infty} u_n(x)$ on $(0,\pi)$, the derivative of the sum equals to the sum of the derivatives $u_n'(x)$?

   (a) $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$
   (b) $\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$
   (c) $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$.

6. Prove that
$$\int_0^1 x^{-x}dx = \sum_{n=1}^{\infty} \frac{1}{n^n}.$$
   [Hints: (a) Note that $x^{-x} = e^{-x\ln x} = \sum_{n=0}^{\infty} \frac{(-1)^n (x\ln x)^n}{n!}$ and prove that this series converges uniformly on $[0,1]$. (b) Use integration by parts and induction to prove that $\int_0^1 x^n(\ln x)^n dx = (-1)^n \frac{n!}{(n+1)^{n+1}}$.]