EXERCISES 4.2 (submit by 08.05.2015)

All the vector functions in these exercises take values in $\mathbb{R}^3$.

1. Let $r_1(t)$ and $r_2(t)$ be vector function defined in a neighborhood of t_0, and there exist limits $\lim_{t\to t_0} r_1(t)$ and $\lim_{t\to t_0} r_2(t)$. Prove that

   (a) $\lim_{t\to t_0} r_1(t)\cdot r_2(t) = (\lim_{t\to t_0} r_1(t))\cdot(\lim_{t\to t_0} r_2(t))$,

   (b) $\lim_{t\to t_0} r_1(t)\times r_2(t) = (\lim_{t\to t_0} r_1(t))\times(\lim_{t\to t_0} r_2(t))$,

   where $r_1(t)\cdot r_2(t)$ is the scalar product of $r_1(t)$ and $r_2(t)$, and $r_1(t)\times r_2(t)$ is the vector product of $r_1(t)$ and $r_2(t)$.

2. Let $r_1(t)$ and $r_2(t)$ be differentiable vector functions. Prove that

$$(r_1(t)\cdot r_2(t))' = r_1'(t)\cdot r_2(t) + r_1(t)\cdot r_2'(t).$$

3. Let $r(t)$ be a differentiable vector function defined on $[a,b]$. Let $\|r(t)\| = 1$ for all $t\in[a,b]$. Prove that the vectors $r(t)$ and $r'(t)$ are orthogonal for all $t\in(a,b)$, i.e.,

$$r(t)\cdot r'(t) = 0.$$

   This has a simple geometric interpretation: the velocity of a particle moving on a sphere is always orthogonal to the radius. [Hint: Differentiate $\|r(t)\|^2$ using previous exercise.]

4. Let $r(t)$ be a differentiable vector function on $[a,b]$, and $r(t)\neq 0$ for all t. Let $r_0(t) = \frac{r(t)}{\|r(t)\|}$ be the radial unit vector. Prove that

$$r'(t) = (r_0(t)\cdot r'(t))\, r_0(t) + \|r(t)\| r_0'(t).$$

   Note that by the previous exercise $r_0(t)$ is orthogonal to $r_0'(t)$. Thus, the formula above is the orthogonal decomposition of the velocity $r'(t)$ into radial ($(r_0(t)\cdot r'(t))\, r_0(t)$) and transverse ($\|r(t)\| r_0'(t)$) components.