EXERCISES 4.1 (submit by 08.05.2015)

1. Let $\langle S, \cdot \rangle$ be a semigroup. Let $a, b \in S$ satisfy $a \cdot b = b \cdot a$. Prove that for all $n \geq 1$, $(a \cdot b)^n = a^n \cdot b^n$, where a^n is defined recursively by $a^1 = a$, and $a^n = a^{n-1} \cdot a$.
2. Let A and B be subgroups of $\langle G, \cdot \rangle$. Prove that $A \cap B$ is also a subgroup of $\langle G, \cdot \rangle$.
3. Let $SL(2, \mathbb{Z})$ be the set $\{M \in M_{n,n}(\mathbb{Z}) : \det(M) = 1\}$ of matrices with integer entries and determinant equal to 1 with the product operation "$\cdot$". Prove that $SL(2, \mathbb{Z})$ is a group. Prove that it is infinite.
4. Let $\mathbb{R}_+$ denote the set of positive real numbers. Prove that the group $\langle \mathbb{R}_+, \cdot \rangle$ is isomorphic to $\langle \mathbb{R}, + \rangle$. [Hint: Use the map $f(x) = \ln x$.]
5. Prove that every element of $SO(3)$ is a rotation in the space $\mathbb{R}^3$ around some line. [Hint: Use canonical form of orthogonal operators on $\mathbb{R}^3$.]