EXERCISES 3.1 (submit by 01.05.2015)

In all the exercises, the space is a finite-dimensional inner product space over $\mathbb{C}$.

1. Two matrices $A, B \in M_n(\mathbb{C})$ are *unitary equivalent* if there exists a unitary matrix $C \in M_n(\mathbb{C})$ such that $B = C^{-1}AC$.

   (a) Let $A, B \in M_n(\mathbb{C})$ be unitary equivalent. Prove that

   $$\sum_{i=1}^{n}\sum_{j=1}^{n}|A_{ij}|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}|B_{ij}|^2.$$

   (b) Prove that two normal matrices $A, B \in M_n(\mathbb{C})$ are unitary equivalent if and only if their characteristic polynomials coincide, i.e., $P_A(\lambda) = P_B(\lambda)$ for all $\lambda \in \mathbb{C}$.

2. Let T be a unitary operator on V. Prove that all eigenvalues of T have modulus 1. Prove that $|\det A_T| = 1$.

3. Prove the following properties of non-negative operators:

   (a) If T_1, T_2 are non-negative operators, α, β are non-negative real numbers, then $\alpha T_1 + \beta T_2$ is non-negative.

   (b) For any linear operator T, the operator TT^* is non-negative.

   (c) If T is self-adjoint, then T^2 is non-negative.

   (d) Let T be a self-adjoint operator. Then, T is non-negative if and only if all its eigenvalues are non-negative.

4. Let T be a linear operator. Prove that $(\mathrm{Ker}(T))^{\perp} = \mathrm{Im}(T^*)$.

5. Let T be a self-adjoint operator. Let $\lambda_1 \leq \ldots \leq \lambda_n$ be the eigenvalues of T. Prove that

   $$\lambda_1 = \min_{v \neq 0} \frac{\langle T(v), v\rangle}{\langle v, v\rangle} \qquad \text{and} \qquad \lambda_n = \max_{v \neq 0} \frac{\langle T(v), v\rangle}{\langle v, v\rangle}.$$