EXERCISES 2.1 (submit by 24.04.2015)

In all the exercises, V is a finite-dimensional inner product space.

1. Let S be a subset of V (not necessarily a vector subspace). Prove that
$$\left(S^{\perp}\right)^{\perp} = \operatorname{span}(S).$$

2. Let U be a vector subspace of V. Prove that for any $v_1, v_2 \in V$,
$$\langle P_U(v_1), v_2\rangle = \langle v_1, P_U(v_2)\rangle.$$

3. For $u \in V$, let $U = \operatorname{span}(\{u\})$. Prove that for all $v \in V$, $P_U(v) = \frac{\langle v,u\rangle}{\|u\|^2}u$.

4. For a linear map $T : V \to V$ and a vector $v \in V$, define the map $f_{v,T} : V \to F$ as $f_{v,T}(u) = \langle T(u), v\rangle$, for all $u \in V$. Prove that $f_{v,T}$ is linear.

5. Let $T : V \to V$ be a linear map. Let $R : V \to V$ be a map such that for all $u, v \in V$, $\langle T(u), v\rangle = \langle u, R(v)\rangle$. Prove that R is linear.

6. Let $T, T_1, T_2 : V \to V$ be linear maps. Prove the following properties of adjoints:
   (a) $(T^*)^* = T$,
   (b) for any $\alpha \in F$, $(\alpha T)^* = \overline{\alpha}T^*$,
   (c) $(T_1 + T_2)^* = T_1^* + T_2^*$,
   (d) $(T_1T_2)^* = T_2^*T_1^*$.