EXERCISES 1.2 (submit by 17.04.2015)

In all the exercises, V is a finite-dimensional inner product space.

1. Let $u, v \in V$ be linearly dependent. Prove that $|\langle u, v\rangle| = \|u\| \cdot \|v\|$.

2. Prove that for any real numbers $x_1, \ldots, x_n$,

$$(x_1 + \ldots + x_n)^2 \leq n\left(x_1^2 + \ldots + x_n^2\right).$$

   When is the inequality strict?

3. Let $v \in V$ such that $\langle u, v\rangle = 0$ for all $u \in V$. Prove that $v = \overrightarrow{0}$.

4. Let $u, v \in V$ be orthogonal. Prove that $\|u + v\|^2 = \|u\|^2 + \|v\|^2$ (The Pythagoras theorem).

5. Let $v_1, \ldots, v_n \in V$ be an orthogonal system, i.e., $v_i \perp v_j$ for all $i \neq j$. Prove that $v_1, \ldots, v_n$ are linearly independent.

6. Let S be a subset of V (not necessarily a vector subspace). Define the orthogonal complement of S by

$$S^\perp = \{v \in V \;:\; \langle u, v\rangle = 0 \text{ for all } u \in S\}.$$

   Prove that $S^\perp$ is a vector subspace of V.