EXAM SOLUTIONS, 20.07.2015

1. (4 points) The matrix of a linear operator T on $\mathbb{R}^2$ in the basis $\{(1,0),(0,1)\}$ is $A_T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Is $T \geq 0$? Is $T > 0$? Justify your answers.

   Answer: T is non-negative, but not positive.

   Solution 1. Let $e_1 = (1,0), e_2 = (0,1)$, then $T(e_1) = T(e_2) = (1,1)$. By the linearilty of T, for all $v = (v_1, v_2)$, $T(v) = v_1 T(e_1) + v_2 T(e_2) = (v_1 + v_2, v_1 + v_2)$, and $\langle T(v), v\rangle = (v_1+v_2)v_1 + (v_1+v_2)v_2 = (v_1+v_2)^2 \geq 0$. Thus, $T \geq 0$. Let $v = (1,-1)$. Then $\langle T(v), v\rangle = 0$. Thus, T is not positive. □

   Solution 2. Note that the given basis is orthonormal, and the matrix of T is symmetric in this basis. Thus, T is self-adjoint. In particular, $T \geq 0$ iff all its eigenvalues are non-negative, and $T > 0$ iff all its eigenvalues are strictily positive. The characteristic polynomial of T is $\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$. Its roots are $\lambda_1 = 0$, $\lambda_2 = 2$. Thus, $T \geq 0$, but not > 0. □

2. (4 points) For vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $\mathbb{R}^2$ and $x \in \mathbb{R}$, let

   $$f(u,v) = u_1 v_1 + u_1 v_2 + u_2 v_1 + x u_2 v_2.$$

   Find all x, for which f defines an inner product on $\mathbb{R}^2$.

   Answer: $x > 1$.

   Solution. We need to check that (a) $f(u,u) \geq 0$ for all $u \in \mathbb{R}^2$, and $f(u,u) = 0$ iff $u = (0,0)$, (b) for all $u,v,w \in \mathbb{R}^2$, $f(u+v,w) = f(u,w)+f(v,w)$, (c) for all $u,v \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, $f(\alpha u, v) = \alpha f(u,v)$, and (d) for all $u,v \in \mathbb{R}^2$, $f(u,v) = f(v,u)$.

   The properties (b-d) are satisfied for all $x \in \mathbb{R}$ by the definition of f.

   The property (a) states that the symmetric quadratic form $f(u,u)$ is positive definite. By the Sylvester's criterion, a symmetric quadratic form $Q(u,u) = a_{11}u_1^2 + 2a_{12}u_1u_2 + a_{22}u_2^2$ is positive definite iff $a_{11} > 0$ and $\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} > 0$. Thus, f satisfies (a) iff $\begin{vmatrix} 1 & 1 \\ 1 & x \end{vmatrix} > 0$, which holds iff $x > 1$. □

3. (4 points) Does the limit

   $$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^2}{x^2 + y^2}$$

   exist?

   Answer: no.

Solution. Let $f(x,y) = \frac{x^3+y^2}{x^2+y^2}$. If there exists $L = \lim_{(x,y)\to(0,0)} f(x,y)$, then for any choice of $\alpha = \alpha(t)$ and $\beta = \beta(t)$ with $\lim_{t\to 0}(\alpha(t),\beta(t)) = (0,0)$, the limit $\lim_{t\to 0} f(\alpha(t),\beta(t))$ is the same and equals L.

For $\alpha_1(t) = t$, $\beta_1(t) = 0$, $\lim_{t\to 0} f(\alpha_1(t),\beta_1(t)) = \lim_{t\to 0} \frac{t^3}{t^2} = 0$.

For $\alpha_2(t) = 0$, $\beta_2(t) = t$, $\lim_{t\to 0} f(\alpha_2(t),\beta_2(t)) = \lim_{t\to 0} \frac{t^2}{t^2} = 1$.

Since the two limits are different, the limit of $f(x,y)$ at $(0,0)$ does not exist. $\square$

4. (4 points) Let $f(x,y) = e^{x^2+\sin y}$. Compute f_{xyx}.

   Answer: $e^{x^2+\sin y} \cdot (4x^2+2)\cos y$.

   Solution.

$$\begin{aligned} f_x &= e^{x^2+\sin y} \cdot 2x, \\ f_{xy} &= e^{x^2+\sin y} \cdot 2x \cdot \cos y, \\ f_{xyx} &= e^{x^2+\sin y} \cdot (2x)^2 \cdot \cos y + e^{x^2+\sin y} \cdot 2 \cdot \cos y. \end{aligned}$$

$\square$

5. (4 points) Let $f(x,y) = x^2 y$. Find a unit vector $\ell \in \mathbb{R}^2$ for which $|D_\ell(1,1)|$ is maximal.

   Answer: $(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$ *or* $(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}})$

   Solution 1. Let $\ell = (a,b) \in \mathbb{R}^2$ with $a^2+b^2 = 1$. Then,

$$D_\ell(x,y) = f_x a + f_y b = 2xya + x^2 b,$$

and $D_\ell(1,1) = 2a+b$. By the Cauchy-Schwarz inequality,

$$|2a+b| \le \sqrt{2^2+1^2} \cdot \sqrt{a^2+b^2} = \sqrt{5},$$

with the equality attained for $a = \frac{2}{\sqrt{5}}, b = \frac{1}{\sqrt{5}}$ or $a = -\frac{2}{\sqrt{5}}, b = -\frac{1}{\sqrt{5}}$.

Remark 0.1. Instead on the Cauchy-Schwarz inequality, one could use the method of Lagrange multipliers.

$\square$

Solution 2. It is known that the gradient of f gives the direction in which the change of f is the largest. Thus, $|D_\ell(1,1)|$ is maximal for $\ell = \pm\frac{\nabla f(1,1)}{\|\nabla f(1,1)\|}$. Since, $\nabla f = (2xy, x^2)$, the optimal choice of ℓ is $\ell = \pm\frac{(2,1)}{\sqrt{2^2+1^2}} = \pm(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$. $\square$

6. (4 points) Let $z = x^2+y^2$, where $x = st$ and $y = \frac{s}{t}$. Compute $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

   Answer: $z_s = 2st^2 + 2\frac{s}{t^2}$, $z_t = 2s^2 t - 2\frac{s^2}{t^3}$.

Solution. Using chain rule,

$$\begin{aligned} z_s &= z_x x_s + z_y y_s = 2xt + 2y\frac{1}{t} = 2st^2 + 2\frac{s}{t^2}, \\ z_t &= z_x x_t + z_y y_t = 2xs + 2y(-\frac{s}{t^2}) = 2s^2t - 2\frac{s^2}{t^3}. \end{aligned}$$

$\square$

7. (4 points) Find the maximum and minimum of $z = x^2 - 2x + y^2$ on the circle $x^2 + y^2 = 1$.

   Answer: 3 *and* -1.

   Solution. First note that the circle $x^2 + y^2 = 1$ is a closed and bounded set in $\mathbb{R}^2$, thus z attains its maximum and minimum at some points of the circle.

   To find the maximum and minimum of z, we use the method of Lagrange multipliers. For $\lambda \in \mathbb{R}$, let $F(x, y) = x^2 - 2x + y^2 - \lambda(x^2 + y^2 - 1)$. If z attains its maximum or minimum at (x_*, y_*) on the circle $x^2 + y^2 = 1$, then $F_x(x_*, y_*) = F_y(x_*, y_*) = 0$. Thus, the points where z attains its maximum and minimum, are solutions to the following system of equations for some $\lambda \in \mathbb{R}$:

$$\begin{cases} 2x - 2 - \lambda 2x = 0, \\ 2y - \lambda 2y = 0, \\ x^2 + y^2 = 1. \end{cases}$$

   From the second equation, either $y = 0$ or $\lambda = 1$. By the first equation $\lambda \neq 1$, thus $y = 0$. By the third equation, $x = \pm 1$. Thus, the maximum and minimum of z are attained at $(1, 0)$ and $(-1, 0)$. We compute, $z(1, 0) = -1$, $z(-1, 0) = 3$. $\square$

8. (4 points) Let $s(x) = \sum_{n=1}^{\infty} \frac{x^2}{x^2+n^2}$. Is $s'(x)$ continuous on $\mathbb{R}$?

   Answer: yes.

   Solution. First note that the given series converges for every $x \in \mathbb{R}$, since for each $n \geq 1$, $\frac{x^2}{x^2+n^2} \leq \frac{x^2}{n^2}$, and $\sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$.

   Let $u_n(x) = \frac{x^2}{x^2+n^2}$. We first show that $s(x)$ is differentiable for all $x \in \mathbb{R}$, and $s'(x) = \sum_{n=1}^{\infty} u_n'(x)$. For this, it suffices to show that for every $a > 0$, the series $\sum_{n=1}^{\infty} u_n'(x)$ converges uniformly on $[-a, a]$. We compute

$$u_n'(x) = \frac{2x \cdot (x^2 + n^2) - x^2 \cdot 2x}{(x^2 + n^2)^2} = \frac{2xn^2}{(x^2 + n^2)^2}.$$

   Thus,

$$\sup_{x \in [-a, a]} |u_n'(x)| \leq \frac{2an^2}{n^4} = \frac{2a}{n^2}.$$

Since the series $\sum_{n\geq 1} \frac{2a}{n^2}$ converges, by the Weierstrass theorem (M-test), the series $\sum_{n=1}^{\infty} u_n'(x)$ converges uniformly on $[-a, a]$ for any choice of $a > 0$. We conclude that $s(x)$ is differentiable for all $x \in \mathbb{R}$, and

$$s'(x) = \sum_{n=1}^{\infty} \frac{2xn^2}{(x^2+n^2)^2}.$$

Finally, notice that the functions $u_n'(x) = \frac{2xn^2}{(x^2+n^2)^2}$ are continuous on $\mathbb{R}$ (and, in particular, on $[-a, a]$ for any $a > 0$). Thus, for any $a > 0$, $s'(x)$ is the sum of a uniformly convergent series of continuous functions, which implies that $s'(x)$ is also continuous on $[-a, a]$. Since $a > 0$ is arbitrary, $s'(x)$ is continuous on $\mathbb{R}$. $\square$

9. (4 points) Is $f(x, y) = \left(\sqrt{|x|} + \sqrt{|y|}\right)^2$ a norm on $\mathbb{R}^2$?

Answer: no.

Solution. The function f defines a norm on $\mathbb{R}^2$ if (a) for all $(x, y) \in \mathbb{R}^2$, $f(x, y) \geq 0$ and $f(x, y) = 0$ iff $(x, y) = (0, 0)$, (b) for all $(x, y) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$, $f(\lambda x, \lambda y) = |\lambda| f(x, y)$, (c) for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, $f(x_1+x_2, y_1+y_2) \leq f(x_1, y_1)+f(x_2, y_2)$.

Properties (a-b) are immediate from the definition of f. However, property (c) is violated. Indeed, let $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (0, 1)$, then $f(x_1 + x_2, y_1 + y_2) = f(1, 1) = 4$, and $f(x_1, y_1) + f(x_2, y_2) = 1 + 1 = 2$. $\square$

10. (4 points) Find the general solution to $(x + y)^2 y' = 1$.

Answer: $y = \arctan(x + y) + C$ *or, equivalently,* $x + y = \tan(y - C)$.

Solution. This is an equation of the form $y' = f(x+y)$, thus we make a substitution $z = x + y$. Then, $dz = dx + dy$, and the equation becomes $z^2(dz - dx) = dx$, or, equivalently,

$$\frac{z^2 dz}{1 + z^2} = dx.$$

Integration gives

$$z - \arctan z = x + C,$$

which in the original variables gives the general solution $y = \arctan(x + y) + C$, or, equivalently, $x + y = \tan(y - C)$. $\square$

11. (4 points) Find the general solution to $y'' - y = 2x$.

Answer: $y = -2x + C_1 e^x + C_2 e^{-x}$.

Solution. This is a linear equation, thus its general solution equals to the sum of its particular solution and the general solution to the homogeneous equation $y'' - y = 0$.

It is easy to see that $y_1 = -2x$ is a particular solution to $y'' - y = 2x$. The general solution to $y'' - y = 0$ is $y_2 = C_1 e^x + C_2 e^{-x}$. Thus, the general solution to $y'' - y = 2x$ is $y = -2x + C_1 e^x + C_2 e^{-x}$. $\square$