RETAKE SOLUTIONS, 07 October 2016

1. (4 points) Let v_1, v_2, v_3 be linearly independent vectors. Are the vectors

$$w_1 = v_1 + v_2 - v_3, \quad w_2 = v_2 + v_3 - v_1, \quad w_3 = v_3 + v_1 - v_2$$

linearly independent?

Answer: Yes.

Solution. The vectors w_1, w_2, w_3 are linearly independent if and only if $\det\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \neq 0$. Since this determinant equals to $4 \neq 0$, the vectors are linearly independent. □

2. (4 points) Let (e_1, e_2) be an orthonormal basis of a Euclidean vector space V. Let $T \in \mathcal{L}(V, V)$ such that its matrix in the basis $(e_1, e_1 + e_2)$ is $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Find the matrix of T^* in the basis $(e_1, e_1 + e_2)$.

Answer: $\begin{pmatrix} 4 & 7 \\ -2 & -4 \end{pmatrix}$.

Solution. Let $f_1 = e_1, f_2 = e_1 + e_2$. The catch here is that the basis (f_1, f_2) is *not* orthonormal. Thus, we first find the matrix of T in the basis (e_1, e_2). Denote by $A_T^{(e)}$ the matrix of T in the basis (e_1, e_2) and by $A_T^{(f)}$ its matrix in the basis (f_1, f_2). Let the transition matrix from (e_1, e_2) to (f_1, f_2) be $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then

$$A_T^{(e)} = C\, A_T^{(f)}\, C^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 1 & -2 \end{pmatrix}.$$

Then the matrix of T^* in the basis (e_1, e_2) is $A_{T^*}^{(e)} = \left(A_T^{(e)}\right)^t = \begin{pmatrix} 2 & 1 \\ -2 & -2 \end{pmatrix}$. Finally, the matrix of T^* in the basis (f_1, f_2) is

$$A_{T^*}^{(f)} = C^{-1}\, A_{T^*}^{(e)}\, C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ -2 & -4 \end{pmatrix}.$$

□

3. (4 points) The matrix of a linear operator T in some orthonormal basis is $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$. Is T positive definite?

Answer: No.

Solution. Since the matrix of T in an orthonormal basis is not symmetric, T is not self-adjoint. Thus, it is not positive definite.

Despite that, for any $v \neq 0$, $\langle T(v), v\rangle > 0$! ☐

4. (4 points) For which values of $\lambda \in \mathbb{R}$ the following quadratic form on $\mathbb{R}^3$ is positive definite?

$$Q(x) = 5x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 - 2x_1x_3 - 2\lambda x_2x_3$$

Answer: $(0, \frac{4}{5})$.

Solution. The matrix of the corresponding symmetric bilinear form is

$$A = \begin{pmatrix} 5 & 2 & -1 \\ 2 & 1 & -\lambda \\ -1 & -\lambda & 1 \end{pmatrix}.$$

By Sylvester's criterion, Q is positive definite if and only if all the leading principal minors of A are positive. We have

$$|5| = 5 > 0, \quad \begin{vmatrix} 5 & 2 \\ 2 & 1 \end{vmatrix} = 1 > 0, \quad |A| = \lambda(4 - 5\lambda) > 0 \text{ iff } \lambda \in \left(0, \frac{4}{5}\right).$$

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5. (4 points) Is the following function continuous on $\mathbb{R}^2$?

$$f(x, y) = \begin{cases} \frac{x^2y}{x^4+y^2} & \text{if } x^2 + y^2 > 0, \\ 0 & \text{if } x = y = 0. \end{cases}$$

Answer: No.

Solution. The function f is continuous at every $(x, y) \neq (0, 0)$ as the ratio of a polynomial and a positive polynomial, but it is not continuous at $(0, 0)$. Indeed, if $\alpha(t) = t, \beta(t) = t^2$, then $\lim_{t\to 0} f(\alpha(t), \beta(t)) = \frac{1}{2} \neq 0 = f(0, 0)$.

In fact, even the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist, since for $\alpha(t) = t, \beta(t) = 0$, $\lim_{t\to 0} f(\alpha(t), \beta(t)) = 0 \neq \frac{1}{2}$. ☐

6. (4 points) Let $f(x, y) = e^{\sin(x+y)}\, xy$. Compute $\frac{\partial^2 f}{\partial x \partial y}(0, 0)$.

Answer: 1.

Solution. We first compute $\frac{\partial f}{\partial y} = e^{\sin(x+y)} \cos(x+y)\, xy + e^{\sin(x+y)}\, x$. Thus, $\frac{\partial f}{\partial y}(x, 0) = e^{\sin x}\, x$. Then

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) = \frac{d}{dx}\left(e^{\sin x}\, x\right)(0) = \left(e^{\sin x} \cos x\, x + e^{\sin x}\right)\Big|_{x=0} = 1.$$

☐

7. (4 points) Find the maximum and minimum of the function $u(x, y) = 3 + 2xy$ on the set $x^2 + y^2 \le 1$.

Answer: 4 *and* 2.

Solution. We first find local extrema of u in the set $x^2 + y^2 < 1$. We compute $u_x = 2y$ and $u_y = 2x$. Thus, $u_x = u_y = 0$ only at the point $(0, 0)$. At this point, $u(0, 0) = 3$.

It remains to find extrema of u on the boundary of the disc, $x^2 + y^2 = 1$. We use the method of Lagrange multipliers. Let $F(x, y, \lambda) = 3+2xy-\lambda(x^2+y^2-1)$. Then $F_x = 2y - 2\lambda x$ and $F_y = 2x - 2\lambda y$. Thus, if $F_x = F_y = 0$, then $y = \lambda^2 y$, which means that either $y = 0$ or $\lambda^2 = 1$.

Consider first the case $y = 0$. From the constraint we find $x = \pm 1$, and $u(\pm 1, 0) = 3$. Next, consider the case $y \ne 0$, $\lambda^2 = 1$. In this case either $x = y = \pm\frac{1}{\sqrt{2}}$ and $u(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}) = 4$, or $x = -y = \pm\frac{1}{\sqrt{2}}$ and $u(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}) = 2$. Thus, the maximum of u is 4 and the minimim 2. □

8. (4 points) Does the sequence of functions $f_n(x) = n \sin\frac{1}{nx}$ converge uniformly on $[1, +\infty)$ as $n \to \infty$?

Answer: Yes.

Solution. We first compute the pointwise limit of $f_n(x)$. For each $x \ge 1$,

$$f_n(x) = n \sin\frac{1}{nx} = \frac{\sin\frac{1}{nx}}{\frac{1}{nx}} \cdot \frac{1}{x} \to \frac{1}{x}, \quad \text{as } n \to \infty.$$

Thus, if f_n converges uniformly on $[1, +\infty)$, then it must converge to $\frac{1}{x}$. To prove the uniform convergence, we need to show that $\sup_{x\ge 1} \left|f_n(x) - \frac{1}{x}\right| \to 0$ as $n \to \infty$. To find this supremum, we compute

$$\left(f_n(x) - \frac{1}{x}\right)' = \frac{1}{x^2}\left(1 - \cos\frac{1}{nx}\right).$$

Since for $x \ge 1$, $\frac{1}{nx} \in (0, 2\pi)$, the above derivative is always positive. Thus, the supremum is either attained at $x = 1$ or at infinity. It is easy to see that it is attained at $x = 1$:

$$\sup_{x\ge 1}\left|f_n(x) - \frac{1}{x}\right| = |f_n(1) - 1| = 1 - n\sin\frac{1}{n} > 0.$$

Since the right hand side tends to 0 as $n \to \infty$, we conclude that $f_n(x)$ converge uniformly to $\frac{1}{x}$ on $[1, +\infty)$ as $n \to \infty$. □

9. (4 points) Let $F = \{f \in C[0, 1] \ : \ f(x) = \alpha x^2 \text{ for some } \alpha \in [0, 1]\}$. Is F compact in $C[0, 1]$?

Answer: Yes.

Solution. F is compact if for any sequence of functions $f_n \in F$, one can select a subsequence f_{n_k} such that f_{n_k} converges in the metric of $C[0,1]$ to some function $f \in F$.

Let $f_n \in F$. Then there exist $\alpha_n \in [0,1]$ such that $f_n(x) = \alpha_n x^2$. Since α_n is a bounded sequence of real numbers, there exists a subsequence α_{n_k} and a real number $\alpha \in [0,1]$ such that $\alpha_{n_k} \to \alpha$ as $k \to \infty$. Let $f(x) = \alpha x^2$. Note that $f \in F$. We claim that f_{n_k} converges to f in $C[0,1]$. Indeed,

$$\sup_{x\in[0,1]} |f_{n_k}(x) - f(x)| = \sup_{x\in[0,1]} \left|\alpha_{n_k} x^2 - \alpha x^2\right| = |\alpha_{n_k} - \alpha| \to 0, \quad \text{as } k \to \infty.$$

Since f_n was an arbitrary sequence of functions from F, we conclude that F is compact. □

10. (4 points) Find the general solution to the differential equation $(x + 2y^3)y' = y$.

    Answer: $x = Cy + y^3$ *and* $y = 0$.

    Solution. We change the role of variables, namely assume that $x = x(y)$. Then $x' = \frac{1}{y'}$ and the equation becomes $x' - \frac{1}{y}x = 2y^2$. (Assuming that $y \neq 0$.) This is a linear ODE. We first find the general solution to the homogeneous equation $x' - \frac{1}{y}x = 0$. By separating variables and integrating, we obtain that $x = Cy$. To find a particular solution to the nonhomogeneous equation, we replace in the above solution the constant C by an unknown function of y, $C(y)$, and substitute into the equation:

$$(Cy)' - \frac{1}{y}(Cy) = 2y^2$$

    From this we find $C(y) = y^2 + C$. Thus, the general solution to the ODE is $x = Cy + y^3$.

    In the above calculation we assumed that $y \neq 0$. A substitution in the equation shows that $y = 0$ is also a solution to the ODE. □

11. (4 points) Find the general solution to the differential equation $y'' - 2y' + y = 2e^x$.

    Answer: $y = (C_1 + C_2 x + x^2)e^x$.

    Solution. This is a linear ODE with constant coefficients. We first find the general solution to the homogeneous equation $y'' - 2y' + y = 0$. The characteristic equation $\lambda^2 - 2\lambda + 1 = 0$ has the root $\lambda = 1$ of multiplicity 2. Thus, the general solution is $y_1(x) = (C_1 + C_2 x)e^x$.

    Next, we search for a particular solution to the nonhomogeneous equation in the form $y_2(x) = ax^2 e^x$. A substitution in the equation gives $a = 1$. Thus, the general solution to the nonhomogeneous ODE is $y(x) = y_1(x) + y_2(x) = (C_1 + C_2 x + x^2)e^x$. □