EXERCISES 6.1 (submit by 20.05.2016)

1. Consider the Euclidean space $\mathbb{R}^3$. Let
$$B(x,y) = (x \times y)_1 + (x \times y)_2 + (x \times y)_3, \quad x,y \in \mathbb{R}^3,$$
i.e., $B(x,y)$ is the sum of the coordinates of the cross product of x and y.
   (a) Prove that B is a bilinear form on $\mathbb{R}^3$.
   (b) What is its matrix in the canonical basis of $\mathbb{R}^3$.
   (c) Is B symmetric?
   (d) Is B non-degenerate?

2. Consider the vector space of square matrices $M_n(\mathbb{C})$ over $\mathbb{C}$. For $A \in M_n(\mathbb{C})$, the *trace* of A is the scalar $\mathrm{tr}(A) = \sum_{i=1}^n A_{ii}$. Let
$$B(A,B) = \mathrm{tr}(A\overline{B}), \quad A, B \in M_n(\mathbb{C}).$$
   (a) Prove that B is a sesquilinear form on $M_n(\mathbb{C})$.
   (b) What is its matrix in the basis $(E_{ij})_{1\le i,j\le n}$ of $M_n(\mathbb{C})$? The elements of matrix E_{ij} are $(E_{ij})_{kl} = \begin{cases} 1 & k=i, l=j \\ 0 & \text{else.} \end{cases}$
   (c) Is B Hermitian?
   (d) Is B non-degenerate?

3. Assume that the matrix of a bilinear form B in a basis (e_1, e_2, e_3) is $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Consider the new basis $e_1' = e_1 - e_2, \quad e_2' = e_1 + e_3, \quad e_3' = e_1 + e_2 + e_3$. Find the matrix of B in the new basis.

4. Let B be a sesquilinear form on a vector space over $\mathbb{C}$. Find $B(u,v)$ if the matrix of B in some basis is $\begin{pmatrix} 5 & 2 \\ -1 & i \end{pmatrix}$ and vectors u and v in this basis have coordinates $u = (i, -2)$, $v = (1-i, 3+i)$.

5. For which values of λ the following quadratic form on $\mathbb{R}^3$ is positive definite:
$$Q(x) = 5x_1^2 + x_2^2 + \lambda x_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3.$$

6. Let $r > 0$ and $S_r = \{z \in \mathbb{C} \, : \, |z| \le r\}$. For which r the set S_r with the multiplication operation is a (a) semigroup, (b) group?

7. An isomorphism of a group $\langle G, \cdot \rangle$ to itself is called an automorphism. For which groups G the map $f : G \to G$ defined by $f(x) = x^{-1}$ is an automorphism.

8. Which of the following sets of real n-by-n matrices form a ring with respect to addition and multipication operations on matrices: (a) symmetric matrices, (b) orthogonal matrices, (c) upper triangular matrices (A is upper triangular if $A_{ij} = 0$ for all $i > j$).