EXERCISES 5.1 (submit by 13.05.2016)

In all the exercises, V is a finite-dimensional inner product space.

1. Prove that an operator T is normal if and only if $\|T(v)\| = \|T^*(v)\|$ for all $v \in V$.

2. Let T be a self-adjoint operator. Let $\lambda_1 \le \ldots \le \lambda_n$ be the eigenvalues of T. Prove that
$$\lambda_1 = \min_{v\neq 0} \frac{\langle T(v), v\rangle}{\langle v, v\rangle} \qquad \text{and} \qquad \lambda_n = \max_{v\neq 0} \frac{\langle T(v), v\rangle}{\langle v, v\rangle}.$$

3. Assume that the matrix of an operator $T \in \mathcal{L}(\mathbb{R}^3)$ in some orthonormal basis is $\begin{pmatrix} 13 & 14 & 4 \\ 14 & 24 & 18 \\ 4 & 18 & 29 \end{pmatrix}$. What is the matrix of the positive self-adjoint operator S such that $S^2 = T$ in this basis?

4. Let V be Euclidean (or unitary) space, $x, y \in V$. Prove that the following claims are equivalent:
   (a) there exists an orthogonal (or unitary) operator T such that $T(x) = y$,
   (b) $\|x\| = \|y\|$.

5. Let T be a unitary operator on V. Prove that all eigenvalues of T have modulus 1. Prove that $|\det A_T| = 1$.

6. Consider the orthogonal operator T in $\mathbb{R}^3$ with the matrix
$$A_T = \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}.$$
   Find the canonical form of T and the corresponding orthonormal basis.

7. Let T be a self-adjoint operator, $a, b \in \mathbb{R}$. Prove that the following claims are equivalent:
   (a) $\mathrm{Spec}(T) \subseteq [a, b]$,
   (b) the operator $(T - \lambda\,\mathrm{Id})$ is positive for $\lambda < a$ and negative for $\lambda > b$. (An operator S is negative if the operator $-S$ is positive.)

8. Let T be a linear operator. Prove that $(\mathrm{Ker}(T))^\perp = \mathrm{Im}(T^*)$.