**EXERCISES 3.1** (submit by 29.04.2016)

1. Consider the following two collections of vectors in $\mathbb{R}^3$:

$$e_1 = (1,2,1), \quad e_2 = (2,3,3), \quad e_3 = (3,8,2),$$

and

$$e_1' = (3,5,8), \quad e_2' = (5,14,13), \quad ,e_3' = (1,9,2).$$

Prove that each collection is a basis of $\mathbb{R}^3$. Find the transition matrix from the first to the second basis.

2. How the transition matrix from one basis to another will change if

   (a) two vectors in the first basis are swapped,

   (b) two vectors in the second basis are swapped,

   (c) vectors in the both bases are written in the reverse order.

3. Let $T$ be a linear map from $V$ to $W$ such that its matrix in the bases $(e_1, e_2, e_3)$ of $V$ and $(f_1, f_2)$ of $W$ is $\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix}$. Find the matrix of $T$ in the bases $(e_1, e_1 + e_2, e_1 + e_2 + e_3)$ of $V$ and $(f_1, f_1 + f_2)$ of $W$.

4. Let $T$ be a linear operator on $\mathbb{R}^3$ such that for each $x = (x_1, x_2, x_3)$, $T(x) = (x_1, x_1 + 2x_2, x_2 + 3x_3)$. Find the matrix of $T$ in the canonical basis of $\mathbb{R}^3$.

5. Let $V$ be a vector space over a field $F$ and $T \in \mathcal{L}(V,V)$. Prove that $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$ are invariant vector subspaces of $V$ for $T$.

6. Let $V$ be a vector space over a field $F$. Prove that for any vector subspace $U$ of $V$ there exists a vector subspace $U'$ of $V$ such that $V = U \oplus U'$.

7. Let $T : V \to F$ be a non-zero linear functional (i.e., $T(v) \neq 0$ for some $v \in V$) and $U = \mathrm{Ker}(T)$. Prove that $V = U \oplus \mathrm{Span}(\{w\})$ for any $w \notin U$.

8. Let $T$ be a linear operator on a vector space $V$ over a field $F$. Let $e_1, \ldots, e_m$ be a basis of $\mathrm{Ker}(T)$ and $e_{m+1}, \ldots, e_n$ a basis of $\mathrm{Im}(T)$. How does the matrix of $T$ in the basis $e_1, \ldots, e_n$ of $V$ look like?

9. Let $T$ be a linear operator on the vector space $M_n(\mathbb{R})$ of $n$-by-$n$ matrices with real coefficients such that for each $X \in M_n(\mathbb{R})$, $T(X) = X^t$ (here $X^t$ is the transpose of $X$, $(X^t)_{ij} = X_{ji}$). Find the spectrum of $T$ and all eigenvectors.

10. Find the eigenvalues and corresponding eigenvectors of a linear operator $T$ which (in some basis) is described by the matrix $\begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix}$. Is $T$ diagonalizable?

11. Let $T$ be an invertible linear operator on a vector space $V$. Let $U$ be an invariant vector subspace of $V$ for $T$. Prove that $U$ is also invariant for $T^{-1}$.