EXERCISES 2.1 (submit by 22.04.2016)

1. Let A be an n-by-n matrix, $A_1, \dots, A_n$ be its rows and $A^{(1)}, \dots, A^{(n)}$ its columns. Let $\det(A) = d$. Find the determinants of the following matrices as functions of d:
   (a) $\det(A^{(2)}, A^{(3)}, \dots, A^{(n)}, A^{(1)})$, determinant of the matrix obtained from A by moving the first column to the right end of the matrix.
   (b) $\det(A_n, \dots, A_1)$, determinant of the matrix obtained from A by putting its rows in the reverse order.

2. Let A be an n-by-n matrix, c a scalar. How will the determinant of A change if
   (a) each element a_{ij} is multiplied by c^{i-j} (for all indices i and j)?
   (b) the matrix is "rotated" by 90° counterclockwise. [Hint: compare the resulting matrix with the transpose of A.]

3. The numbers 20604, 53227, 25755, 20927, and 289 are all divisible by 17. Prove that the determinant of the following matrix is also divisible by 17 (without computing explicitly the value of the determinant!).

$$\begin{pmatrix} 2 & 0 & 6 & 0 & 4 \\ 5 & 3 & 2 & 2 & 7 \\ 2 & 5 & 7 & 5 & 5 \\ 2 & 0 & 9 & 2 & 7 \\ 0 & 0 & 2 & 8 & 9 \end{pmatrix}$$

4. Let $f_{ij}(x)$ be differentiable functions on $\mathbb{R}$. Consider the n-by-n matrix F with entries $f_{ij}(x)$. Prove that $\det(F)$ is differentiable on $\mathbb{R}$ and

$$(\det(F))' = \sum_{i=1}^{n} \det \begin{pmatrix} f_{11}(x) & \dots & f_{1n}(x) \\ \dots & \dots & \dots \\ f_{i-1,1}(x) & \dots & f_{i-1,n}(x) \\ f_{i1}(x)' & \dots & f_{in}(x)' \\ f_{i+1,1}(x) & \dots & f_{i+1,n}(x) \\ \dots & \dots & \dots \\ f_{n1}(x) & \dots & f_{nn}(x) \end{pmatrix}.$$

5. Using elementary operations with rows and columns, compute the determinant of

$$\begin{pmatrix} 1 & 10 & 100 & 1000 & 10000 & 100000 \\ 0.1 & 2 & 30 & 400 & 5000 & 60000 \\ 0 & 0.1 & 3 & 60 & 1000 & 15000 \\ 0 & 0 & 0.1 & 4 & 100 & 2000 \\ 0 & 0 & 0 & 0.1 & 5 & 150 \\ 0 & 0 & 0 & 0 & 0.1 & 6 \end{pmatrix}.$$

6. Consider the matrices $B \in M_m(F)$, $C \in M_{n-m}(F)$, $D \in M_{m,n-m}(F)$. Let $A \in M_n(F)$ be the block matrix $\begin{pmatrix} B & D \\ 0 & C \end{pmatrix}$. (Here 0 is the $(n-m)$-by-m matrix with all entries being 0's.) Prove that $\det(A) = \det(B) \cdot \det(C)$.