EXERCISES 11.1 (submit by 24.06.2016)

1. Prove that $\rho(f,g) = \max_{x\in[a,b]} |f(x)-g(x)|$ is a metric on $C_{[a,b]}$ (the set of continuous real-valued functions on an interval $[a,b]$).
2. Prove that $\rho(f,g) = \int_a^b |f(x)-g(x)|dx$ is a metric on $C_{[a,b]}$.
3. Let S, S_1, S_2 be subsets of a metric space (X,ρ). Prove the following properties of the closure: (a) $S \subseteq \overline{S}$, (b) $\overline{\overline{S}} = \overline{S}$, (c) if $S_1 \subseteq S_2$ then $\overline{S}_1 \subseteq \overline{S}_2$, (d) $\overline{S_1 \cup S_2} = \overline{S}_1 \cup \overline{S}_2$.
4. Prove that S is open in a metric space (X,ρ) if and only if $X \setminus S$ is closed.
5. Let $\{C_i\}_{i\geq 1}$ be closed sets in (X,ρ). Prove that (a) $\cap_{i=1}^\infty C_i$ is closed, (b) for any $n \geq 1$, $\cup_{i=1}^n C_i$ is closed, (c) $\cup_{i=1}^\infty C_i$ is in general not closed.
6. Let $\{O_i\}_{i\geq 1}$ be open sets in (X,ρ). Prove that (a) $\cup_{i=1}^\infty O_i$ is open, (b) for any $n \geq 1$, $\cap_{i=1}^n O_i$ is open, (c) $\cap_{i=1}^\infty O_i$ is in general not open.
7. Prove that ℓ_2 is complete.
8. Let f be a continuous function from a metric space (X,ρ) to $\mathbb{R}$. Prove that for any $a < b$, the set $f^{-1}((a,b)) = \{x \in X \ : \ f(x) \in (a,b)\}$ is open.
9. Prove that the norms $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ are equivalent in $\mathbb{R}^d$.
10. Prove that the set of all continuous functions on $[0,1]$ equipped with the norm $\|f\| = \int_0^1 |f(x)|dx$ is not a Banach space.
11. Prove that the set of all continuous functions on $[0,1]$ equipped with the norm $\|f\| = \max_{x\in[0,1]} |f(x)|$ is not a Hilbert space, i.e., there exists no inner product $\langle\cdot,\cdot\rangle$ such that for all f, $\langle f,f\rangle = \|f\|^2$.