EXERCISES 1.1 (submit by 15.04.2016)

1. Let $v_1, v_2, \ldots$ be linearly independent vectors and $k \in \mathbb{N}$. Are the following collections of vectors linearly independent?
   (a) $w_1 = 3v_1 + 2v_2 + v_3 + v_4$, $w_2 = 2v_1 + 5v_2 + 3v_3 + 2v_4$, $w_3 = 3v_1 + 4v_2 + 2v_3 + 3v_4$.
   (b) $w_1 = v_1$, $w_2 = v_1 + v_2$, $\ldots$, $w_k = v_1 + v_2 + \ldots + v_k$.
   (c) $w_1 = v_1 + v_2$, $w_2 = v_2 + v_3$, $\ldots$, $w_{k-1} = v_{k-1} + v_k$, $w_k = v_k + v_1$.
   (d) $w_1 = v_1 - v_2$, $w_2 = v_2 - v_3$, $\ldots$, $w_{k-1} = v_{k-1} - v_k$, $w_k = v_k - v_1$.

2. For each of the given collections of vectors, identify a largest subcollection of linearly independent vectors and express all the remaining vectors as linear combinations of the vectors in the subcollection.
   (a) $v_1 = (5, 2, -3, 1)$, $v_2 = (4, 1, -2, 3)$, $v_3 = (1, 1, -1, -2)$, $v_4 = (3, 4, -1, 2)$, $v_5 = (7, -6, -7, 0)$.
   (b) $v_1 = (2, -1, 3, 5)$, $v_2 = (4, -3, 1, 3)$, $v_3 = (3, -2, 3, 4)$, $v_4 = (4, -1, -15, 17)$.
   (c) $v_1 = (4, 3, -1, 1, -1)$, $v_2 = (2, 1, -3, 2, -5)$, $v_3 = (1, -3, 0, 1, -2)$, $v_4 = (1, 5, 2, -2, 6)$.

3. Find the rank of the following matrices.
$$(a) \begin{pmatrix} 8 & 2 & 2 & -1 & 1 \\ 1 & 7 & 4 & -2 & 5 \\ -2 & 4 & 2 & -1 & 3 \end{pmatrix}, \qquad (b) \begin{pmatrix} -6 & 4 & 8 & -1 & 6 \\ -5 & 2 & 4 & 1 & 3 \\ 7 & 2 & 4 & 1 & 3 \\ 2 & 4 & 8 & -7 & 6 \\ 3 & 2 & 4 & -5 & 3 \end{pmatrix}.$$

4. Find the rank of the following matrix for various values of $\lambda \in \mathbb{R}$.
$$\begin{pmatrix} 3 & 4 & 2 & 2 \\ 3 & 17 & 7 & 1 \\ 1 & 10 & 4 & \lambda \\ 4 & 1 & 1 & 3 \end{pmatrix}.$$

5. Prove that for any two matrices A and B for which the product AB is defined, $\mathrm{rank}(AB) \le \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}$.

6. Are the following systems of linear equations consistent? If yes, find the general solution.
$$(a) \begin{cases} 5x_1 + 3x_2 + 5x_3 + 12x_4 = 10, \\ 2x_1 + 2x_2 + 3x_3 + 5x_4 = 4, \\ x_1 + 7x_2 + 9x_3 + 4x_4 = 2. \end{cases} \qquad (b) \begin{cases} -9x_1 + 10x_2 + 3x_3 + 7x_4 = 7, \\ -4x_1 + 7x_2 + x_3 + 3x_4 = 5, \\ 7x_1 + 5x_2 - 4x_3 - 6x_4 = 3. \end{cases}$$

7. Find the inverse to the matrix
$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix}.$$