EXAM, 19 July 2016, 10:00 – 12:00

1. (4 points) Is the matrix $A = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}$ similar to a diagonal matrix? If so, find this diagonal matrix.

   Answer: No.

   Solution. A matrix is similar to a diagonal matrix if and only if for any of its eigenvalues, the algebraic and geometric multiplicities coincide. We first compute the spectrum of A. The characteristic polynomial $\begin{vmatrix} \lambda - 5 & 1 \\ -1 & \lambda - 3 \end{vmatrix} = (\lambda - 4)^2$ has one root $\lambda = 4$ of multiplicity 2. Thus, 4 is the only eigenvalue of A. Its algebraic multiplicity is 2. The geometric multiplicity of 4 is the dimension of the vector space of eigenvectors corresponding to the eigenvalue 4, i.e., all $x = (x_1, x_2)$ which solve $Ax = 4x$. The system of equations $\begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 4 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is satisfied by any $x = (x_1, x_2)$ with $x_1 = x_2$. Thus, the eigenspace of 4 is $\{x \in \mathbb{R}^2 \ : \ x_1 = x_2\}$. Its dimension is 1. Thus, the geometric multiplicity of 4 is 1, which is different from 2. Since the algebraic and geometric multiplicities of 4 are different, the matrix is not diagonalizable. □

2. (4 points) The matrix of a linear operator T on $\mathbb{C}^2$ in the basis $\{(1,0),(0,1)\}$ is $A_T = \begin{pmatrix} 1 & i \\ a & 1 \end{pmatrix}$. For which $a \in \mathbb{C}$ the operator T is normal?

   Answer: $|a| = 1$.

   Solution. An operator T is normal if and only if $T^*T = TT^*$, and if and only if $A_{T^*}A_T = A_T A_{T^*}$. Since the given basis is orthonormal, $A_{T^*} = \overline{A}_T^t = \begin{pmatrix} 1 & \overline{a} \\ -i & 1 \end{pmatrix}$. Thus,

   $$A_T A_{T^*} = \begin{pmatrix} 1 & i \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & \overline{a} \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 2 & \overline{a} + i \\ a - i & |a|^2 + 1 \end{pmatrix}$$

   and

   $$A_{T^*} A_T = \begin{pmatrix} 1 & \overline{a} \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & i \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1 + |a|^2 & i + \overline{a} \\ -i + a & 2 \end{pmatrix}.$$

   Hence, T is normal iff $|a| = 1$. □

3. (4 points) A matrix of a sesquilinear form S in a basis e_1, e_2 is $A_S = \begin{pmatrix} i & 1 - i \\ -1 & 2 + i \end{pmatrix}$. Find $S(x, y)$, where $x = (1, i)$ and $y = (-2i, 1)$ in the basis e_1, e_2.

   Answer: i.

Solution. By the definition, $x = e_1 + ie_2$, $y = -2ie_1 + e_2$. Thus,

$$\begin{aligned} S(x,y) &= S(e_1 + ie_2, -2ie_1 + e_2) = 2iS(e_1,e_1) + S(e_1,e_2) - 2S(e_2,e_1) + iS(e_2,e_2) \\ &= -2 + (1-i) + 2 + i(2+i) = i. \end{aligned}$$

Other way to see this is $S(x,y) = xA_S\overline{y}^t = \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} i & 1-i \\ -1 & 2+i \end{pmatrix} \begin{pmatrix} 2i \\ 1 \end{pmatrix} = i.$ □

4. (4 points) Prove that the following function is continuous on $\mathbb{R}^2$.

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} + \sin(xy) & \text{if } x^2+y^2 > 0 \\ 0 & \text{if } x = y = 0. \end{cases}$$

Solution. First of all note that the function $\sin(xy)$ is continuous on $\mathbb{R}^2$ as a composition of continuous functions. Thus, it suffices to prove that the function

$$g(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & \text{if } x^2+y^2 > 0 \\ 0 & \text{if } x = y = 0 \end{cases}$$

is continuous on $\mathbb{R}^2$. The function $\frac{x^2y^2}{x^2+y^2}$ is continous on $\mathbb{R}^2 \setminus \{(0,0)\}$ as the ratio of polynomials, with denominator never equal to 0. Thus, it suffices to prove that g is continous at $(0,0)$, i.e., $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0$. Note that $0 \le \frac{x^2y^2}{x^2+y^2} \le x^2$. Indeed, the first inequality is obvious, the second inequality is also obvious if $y = 0$, and if $y \neq 0$, then $\frac{x^2y^2}{x^2+y^2} \le \frac{x^2y^2}{y^2} = x^2$. Since $\lim_{(x,y)\to(0,0)} x^2 = 0$, the result follows. □

5. (4 points) Let $f(x,y,z) = \sin(xyz)$. Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$.
Answer: $\cos(xyz)yz$, $-\sin(xyz)xyz^2 + \cos(xyz)z$.

Solution. Using the chain rule, $\frac{\partial f}{\partial x} = \sin'(xyz)\,\frac{\partial}{\partial x}(xyz) = \cos(xyz)\,yz$.
The function f is a composition of sin and a polynomial, hence it is twice continuously differentiable on $\mathbb{R}^2$. Therefore,

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(\cos(xyz)\,yz) = yz\,\frac{\partial}{\partial y}\cos(xyz) + \cos(xyz)\,\frac{\partial}{\partial y}(yz) \\ &= -xyz^2\sin(xyz) + z\cos(xyz). \end{aligned}$$

□

6. (4 points) Let $z = e^{xy}$, where $x = \cos(st)$ and $y = \sin(s+t)$. Find $\frac{\partial z}{\partial t}$.
Answer: $e^{\cos(st)\,\sin(s+t)}\,(\cos(st)\cos(s+t) - s\sin(st)\sin(s+t))$.

Solution. By the chain rule,

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = e^{xy}y\,(-\sin(st)s) + e^{xy}x\,\cos(s+t) \\ &= e^{\cos(st)\,\sin(s+t)}\,(-s\sin(st)\sin(s+t) + \cos(st)\cos(s+t)). \end{aligned}$$

□

7. (4 points) Find maximum and minimum of the function $f(x,y) = x^3 + y^2$ on the ellipse $3x^2 + y^2 = 16$.

   Answer: Maximum $f(0,4) = f(0,-4) = 16$, *minimum* $f(-\frac{4}{\sqrt{3}}, 0) = -\frac{64}{3\sqrt{3}}$.

   Solution. We use the method of Lagrange multipliers. Consider the Lagrangian $F(x,y) = x^3 + y^2 - \lambda(3x^2 + y^2 - 16)$. Every point (x,y) of local extremum of f under the given constraint must satisfy $F_x(x,y) = F_y(x,y) = 0$ for some $\lambda \in \mathbb{R}$. Thus, we have the system of equations

$$\left\{\begin{array}{rcl} 3x^2 - 6\lambda x &=& 0 \\ 2y - 2\lambda y &=& 0 \\ 3x^2 + y^2 &=& 16. \end{array}\right. \iff \left\{\begin{array}{rcl} x(x - 2\lambda) &=& 0 \\ y(1-\lambda) &=& 0 \\ 3x^2 + y^2 &=& 16. \end{array}\right.$$

   We need to consider several cases:

   (a) If $x = 0$, then $y = \pm 4$ and $\lambda = 1$. In this case $f(0, \pm 4) = 16$.

   (b) If $y = 0$, then $x = \pm\frac{4}{\sqrt{3}}$ and $\lambda = \pm\frac{2}{\sqrt{3}}$. In this case $f(\pm\frac{4}{\sqrt{3}}, 0) = \pm\frac{64}{3\sqrt{3}}$.

   (c) If $x \neq 0$ and $y \neq 0$, then $\lambda = 1$, $x = 2$, and $y = \pm 2$. In this case $f(2, \pm 2) = 12$.

   By comparing the above values of f, we conclude that the maximum is 16, attained at points $(0,4)$ and $(0,-4)$, and the minimum is $-\frac{64}{3\sqrt{3}}$ attained at $(-\frac{4}{\sqrt{3}}, 0)$. □

8. (4 points) Compute the limit

$$\lim_{n\to\infty} \int_0^1 (1 - x^2)^{\frac{1}{n}}\, dx.$$

   Justify your answer.

   Answer: 1.

   Solution. Since the sequence of functions $(1-x^2)^{\frac{1}{n}}$ is monotone non-decreasing and bounded from above by 1 on $[0,1]$, the integrals $\int_0^1 (1-x^2)^{\frac{1}{n}}\, dx$ form a non-decreasing bounded sequence in $\mathbb{R}$. Thus, the limit $\lim\limits_{n\to\infty} \int_0^1 (1-x^2)^{\frac{1}{n}}\, dx$ exists and is ≤ 1.

   Let $f_n(x) = (1-x^2)^{\frac{1}{n}}$. Note that $\lim\limits_{n\to\infty} f_n(x) = f(x) = \begin{cases} 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1. \end{cases}$ In particular, the limit is not a continuous function and f_n does not converge uniformly on $[0,1]$. Thus, we cannot directly write that $\lim\limits_{n\to\infty} \int_0^1 f_n(x)\, dx = \int_0^1 f(x)\, dx$.

   However, f_n converges uniformly to f on any interval $[0,\gamma]$, $\gamma \in (0,1)$. Indeed,

$$\sup_{x\in[0,\gamma]} |f_n(x) - f(x)| = \sup_{x\in[0,\gamma]} |(1-x^2)^{\frac{1}{n}} - 1| = 1 - (1-\gamma^2)^{\frac{1}{n}} \to 0, \quad \text{as } n \to \infty.$$

   Thus, $\lim\limits_{n\to\infty} \int_0^\gamma f_n(x)\, dx = \int_0^\gamma f(x)\, dx = \gamma$ for any $\gamma \in (0,1)$.

We conclude that for each $\gamma \in (0,1)$,

$$1 \geq \lim_{n\to\infty} \int_0^1 (1-x^2)^{\frac{1}{n}}\,dx \geq \lim_{n\to\infty} \int_0^\gamma (1-x^2)^{\frac{1}{n}}\,dx = \gamma.$$

Since the inequality holds for all $\gamma \in (0,1)$, we have $\lim_{n\to\infty} \int_0^1 (1-x^2)^{\frac{1}{n}}\,dx = 1$. □

9. (4 points) Let ρ be a metric on X. Prove that $d(x,y) = \min\{\rho(x,y), 1\}$ is also a metric on X.

   Solution. We need to show that d satisfies the axioms of metric:

   (a) For all $x, y \in X$, $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$. Indeed, since $\rho(x,y) \geq 0$, also $d(x,y) \geq 0$, and if $d(x,y) = 0$, then $\rho(x,y) = 0$, which implies $x = y$.

   (b) $d(x,y) = d(y,x)$ for all $x, y \in X$. Immediate from $\rho(x,y) = \rho(y,x)$.

   (c) For all $x, y, z \in V$, $d(x,z) \leq d(x,y) + d(y,z)$. If $d(x,y) = 1$ or $d(y,z) = 1$, then the inequality holds, since $d(x,z) \leq 1$. If $d(x,y) < 1$ and $d(y,z) < 1$, then $d(x,y) = \rho(x,y)$ and $d(y,z) = \rho(y,z)$, and we have $d(x,z) \leq \rho(x,z) \leq \rho(x,y) + \rho(y,z) = d(x,y) + d(y,z)$. □

10. (4 points) Find the general solution to the differential equation $xy' - y = xe^{-\frac{y}{x}}$.

    Answer: $e^{\frac{y}{x}} = \ln|x| + C$, $C \in \mathbb{R}$, $x \in (-\infty, 0)$ *or* $x \in (0, +\infty)$.

    Solution. This is a homogeneous ODE defined for $x \neq 0$. Substitute $y = tx$ ($y' = t'x + t$). In the new variables, the ODE is $xt' = e^{-t}$. By separating variables and integrating, we get $e^t = \ln|x| + C$. Plugging in $t = \frac{y}{x}$ gives the general solution. □

11. (4 points) Find the general solution to the differential equation $y'' + y = 4\cos x$.

    Answer: $y = C_1 \cos x + C_2 \sin x + 2x \sin x$.

    Solution. This is a linear ODE. Its general solution is $y = y_1 + y_2$, where y_1 is the general solution to the homogeneous ODE $y'' + y = 0$, and y_2 is a particular solution to the inhomogeneous ODE.

    We first find y_1. Consider $y'' + y = 0$. The characteristic equation is $\lambda^2 + 1 = 0$ with solutions $\lambda = \pm i$. Thus, $y_1 = C_1 \cos x + C_2 \sin x$.

    It remains to find y_2. The right hand side has the form $P(x)\cos x + Q(x)\sin x$. Since i is a solution to the characteristic equation of multiplicity 1, we search for a particular solution in the form $y_2(x) = x(A\cos x + B\sin x)$, where A, B are some constants that we need to identify. We first compute

    $$\begin{aligned} y_2' &= A\cos x + B\sin x + x(-A\sin x + B\cos x) \\ y_2'' &= -2A\sin x + 2B\cos x + x(-A\cos x - B\sin x). \end{aligned}$$

    Substitution of y_2 into the ODE gives the equation $-2A\sin x + 2B\cos x = 4\cos x$, which is the identity for $A = 0$ and $B = 2$. Thus, $y_2(x) = 2x\sin x$.

    The general solution is $y(x) = C_1 \cos x + C_2 \sin x + 2x \sin x$. □