Exercises 4.2, Mathematics 1 (12-PHY-BIPMA1)
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1. Which of the following functions is differentiable at 0:

$$f(x) = x|x|, \quad g(x) = x\sqrt{|x|}, \quad h(x) = x + |x|,$$

$$k(x) = \begin{cases} x^2 \sin \frac{\pi}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases} \qquad l(x) = \begin{cases} x \sin \frac{\pi}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

2. For which real numbers a and b, the following function is differentiable at 0:

$$f(x) = \begin{cases} ax + b & \text{for } x < 0 \\ x - x^2 & \text{for } x \geq 0. \end{cases}$$

3. For which real numbers a and b, the following function is differentiable at 0:

$$f(x) = \begin{cases} ax + b & \text{for } x < 1 \\ x - x^2 & \text{for } x \geq 1. \end{cases}$$

4. Compute the derivatives of the following functions:

$$f(x) = x \sin x + \cos x, \quad g(x) = \cos^2 x, \quad h(x) = \frac{\cos^2 x}{x},$$

$$k(x) = \frac{5e^x}{1 + \tan x}, \quad l(x) = \frac{1 + e^x}{1 - e^x}, \quad m(x) = e^x \sin x \cos^2 x.$$

5. Hyperbolic sine, cosine, tangent, and cotangent, are defined, respectively, as

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}.$$

Prove the analogues of $\cos^2 x + \sin^2 x = 1$ and $\sin(2x) = 2 \sin x \cos x$ for the hyperbolic sine and cosine:

$$\cosh^2 x - \sinh^2 x = 1, \qquad \sinh(2x) = 2 \sinh x \cosh x.$$

Compute the derivatives of $\sinh x$, $\cosh x$, $\tanh x$, and $\coth x$.