Exercises 2.1, Mathematics 1 (12-PHY-BIPMA1)
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1. Prove that for all $n \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $1 \le k \le n$,

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

2. Use mathematical induction to prove that for all $n \in \mathbb{N}$,

   (a) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$,

   (b) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$,

   (c) for any sequence $(a_k)$ of non-negative real numbers,

$$(1+a_1)(1+a_2)\cdot \ldots \cdot (1+a_n) \ge 1 + a_1 + a_2 + \cdots + a_n,$$

   (d) for any $a, b \in \mathbb{R}$,

$$a^{n+1} - b^{n+1} = (a-b)(a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + ab^{n-1} + b^n),$$

   (e) for any $\theta \ne 0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots$,

$$\cos\theta + \cos 2\theta + \cos 3\theta + \cdots + \cos n\theta = \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta} - \frac{1}{2}.$$

3. Prove that

   (a) $\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

   (b) $\lim_{n\to\infty} \frac{n^2}{2^n} = 0$. (Hint: Show using induction that $2^n \ge n^3$ for all $n \ge 10$.)