Exercises 13.2, Mathematics 1 (12-PHY-BIPMA1)

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1. Let $U$ and $V$ be vector spaces over a field $F$. Let $e_1, \dots, e_n$ be a basis in $U$, and $f_1, \dots f_m$ a basis in $V$. Let $T_1$ and $T_2$ be linear maps from $U$ to $V$ represented by the matrices

$$A_{T_1} = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ & \dots & \\ \alpha_{m1} & \dots & \alpha_{mn} \end{pmatrix}, \qquad A_{T_2} = \begin{pmatrix} \beta_{11} & \dots & \beta_{1n} \\ & \dots & \\ \beta_{m1} & \dots & \beta_{mn} \end{pmatrix}.$$

   Let $\alpha \in F$. Prove that

$$A_{T_1+T_2} = \begin{pmatrix} \alpha_{11}+\beta_{11} & \dots & \alpha_{1n}+\beta_{1n} \\ & \dots & \\ \alpha_{m1}+\beta_{m1} & \dots & \alpha_{mn}+\beta_{mn} \end{pmatrix}, \qquad A_{\alpha T_1} = \begin{pmatrix} \alpha\alpha_{11} & \dots & \alpha\alpha_{1n} \\ & \dots & \\ \alpha\alpha_{m1} & \dots & \alpha\alpha_{mn} \end{pmatrix}.$$

2. Let $F$ be a field. Prove that $M_{m,n}(F) \cong F^{mn}$.

3. Let $F$ be a field. Let $A \in M_{k,l}(F)$, $B \in M_{l,m}(F)$, $C \in M_{m,n}(F)$. Prove that $A(BC) = (AB)C$, i.e., the multiplication of matrices is associative.

4. Let $F$ be a field. Let $A, B \in M_{n,n}(F)$ be ivertible matrices. Prove that
   (a) $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$.
   (b) $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

5. Prove that the set of matrices

$$\left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \ x, y \in \mathbb{R} \right\}$$

   is a vector subspace of $M_{2,2}(\mathbb{R})$. Prove that it is isomorphic to the vector space of complex numbers over $\mathbb{R}$.