Exercises 12.2, Mathematics 1 (12-PHY-BIPMA1)

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1. Let T be an isomorphism between vector spaces V_1 and V_2. Prove that
   (a) $T(\overrightarrow{0_{V_1}}) = \overrightarrow{0_{V_2}}$, i.e., the zero vector of V_1 is mapped to the zero vector of V_2.
   (b) For all $u \in V_1$, $T(-u) = -T(u)$, i.e., additive inverses are mapped to additive inverses.
   (c) If $u_1, \ldots, u_n$ are linearly independent in V_1, then $T(u_1), \ldots, T(u_n)$ are linearly independent in V_2.

2. Let V_1, V_2, V_3 be vector spaces. Prove that
   (a) $V_1 \cong V_1$,
   (b) if $V_1 \cong V_2$, then $V_2 \cong V_1$,
   (c) if $V_1 \cong V_2$ and $V_2 \cong V_3$, then $V_1 \cong V_3$.

   Properties (a)-(c) state that $\cong$ is an *equivalence relation* on vector spaces.

3. Let U_1 and U_2 be vector subspaces of a vector space V.
   (a) Prove that $U_1 \cap U_2$ and $U_1 + U_2$ are vector subspaces of V.
   (b) If U_1 is a vector subspace of U_2, prove that $U_1 \cap U_2 = U_1$ and $U_1 + U_2 = U_2$.