Exercises 12.1, Mathematics 1 (12-PHY-BIPMA1)

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1. Let F be a field. Prove that

   (a) The multiplicative identily 1 is unique in F, i.e., if for some $\alpha \in F$, $\alpha \cdot \beta = \beta$ for all $\beta \in F$, then $\alpha = 1$.

   (b) For any $\alpha \in F$, its additive inverse $-\alpha \in F$ is unique.

   (c) For any $\alpha \in F$, its multiplicative inverse $\alpha^{-1} \in F$ is unique.

2. Let V be a vector space over a field F. Prove that

   (a) For all $\alpha \in F$, $\alpha \cdot \overrightarrow{0} = \overrightarrow{0}$.

   (b) For all $u \in V$, $0 \cdot u = \overrightarrow{0}$.

   (c) For all $u \in V$, $(-1) \cdot u = -u$.

3. Let V be a vector space over a field F. Prove the following claims.

   (a) If a system of vectors U is complete and some vector $u \in U$ can be expressed as a linear combination of the vectors in $U \setminus \{u\}$, then the system of vectors $U \setminus \{u\}$ is also complete.

   (b) If a system of vectors U is linearly independent and some vector $u \notin U$ can not be expressed as a linear combination of the vectors in U, then the system of vectors $U \cup \{u\}$ is also linearly independent.

4. Let e_1, e_2, e_3, e_4 and f_1, f_2, f_3, f_4 be two bases of $\mathbb{R}^4$, and x_1, x_2, x_3, x_4 and y_1, y_2, y_3, y_4 be the coordinates of some vector from $\mathbb{R}^4$ in the first and second bases, respectively. Express the coordinates y_1, y_2, y_3, y_4 as functions of x_1, x_2, x_3, x_4, i.e., compute the coordinate transformation, if the two bases are given by

$$\begin{array}{llll} e_1 = (1, 2, -1, 0), & e_2 = (1, -1, 1, 1), & e_3 = (-1, 2, 1, 1), & e_4 = (-1, -1, 0, 1), \\ f_1 = (2, 1, 0, 1), & f_2 = (0, 1, 2, 2), & f_3 = (-2, 1, 1, 2), & f_4 = (1, 3, 1, 2). \end{array}$$

   [Hint: The equality $y_1 f_1 + y_2 f_2 + y_3 f_3 + y_4 f_4 = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4$ leads to a system of 4 linear equations, where the unknowns are y_i's.]