Exercises 11.1, Mathematics 1 (12-PHY-BIPMA1)
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1. Let e_1, e_2, e_3 be an orthonormal basis. Prove that for any vector u,

$$u = (u, e_1)e_1 + (u, e_2)e_2 + (u, e_3)e_3,$$

i.e., the ith coordinate of u in the basis equals (u, e_i).

2. Prove that $(u + v, w) = (u, w) + (v, w)$.

[Hint: Reduce the problem to the case $|w| = 1$. Consider an orthonormal basis $e_1 = w, e_2, e_3$. Compute the first coordinate of vectors $u + v, u, v$ in this basis.]

3. Prove that

$$(u, v) = \frac{1}{2} \{(u, u) + (v, v) - (u + v, u + v)\}.$$

This identity implies that the following 4 properties define scalar product uniquely:

(a) $(u, v) = (v, u)$, (b) $(u + v, w) = (u, w) + (v, w)$, (c) $(\alpha u, v) = \alpha(u, v)$, (d) $(u, u) = |u|^2$.

4. Let e_1, e_2, e_3 be the right-hand oriented basis. What is the orientation of (a) e_1, e_3, e_2, (b) e_2, e_1, e_3, (c) e_2, e_3, e_1, (d) e_3, e_1, e_2, (e) e_3, e_2, e_1?

5. Prove that $(u + v) \times w = u \times w + v \times w$.

6. Let e_1, e_2, e_3 be an orthonormal basis with right-hand orientation. If u and v are vectors with coordinates $(1, 2, 1)$ and $(1, 1, 0)$, respectively, what are the coordinates of $u \times v$?

7. Prove that

$$(u, v \times w) = -(v, u \times w) = (v, w \times u) = -(w, v \times u) = (w, u \times v) = -(u, w \times v).$$

[Hint: Once the first equality is proved, the other follow either from the anticommutativity of the vector product or from the first equality.]

8. Prove that

$$u \times (v \times w) = (u, w)v - (u, v)w.$$

[Hint: Consider a right-hand oriented orthonormal basis e_1, e_2, e_3 such that (a) the coordinates of w are $(w_1, 0, 0)$, i.e., e_1 is collinear with w, (b) the coordinates of v are $(v_1, v_2, 0)$, i.e., e_2 is coplanar with v and w, and (c) the coordinates of u are (u_1, u_2, u_3). Compute coordinates of $u \times (v \times w)$ and $(u, w)v - (u, v)w$ in this basis.]

9. Prove the Jacobi identity:

$$u \times (v \times w) + v \times (w \times u) + w \times (u \times v) = 0.$$