Syllabus, Mathematics 1 (12-PHY-BIPMA1)

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1. Natural, integer, rational, real numbers. Absolute value. Sets, subsets, union, intersection, difference of sets. Bounded sets. Supremum and infimum of subsets of $\mathbb{R}$. Uniqueness of Sup$A$. Countable and uncountable sets.

2. Sequence of real numbers. Limit of a sequence. Convergent and divergent sequences. ε-neighborhood of a real number. Any sequence has at most one limit. Properties of limits. Bounded sequences. Monotone increasing and monotone decreasing sequences. Convergence of bounded monotone sequences. Subsequences. Bolzano-Weierstrass theorem. Cauchy sequence. Cauchy criterion for convergence. Infinite limits. Cluster point. Upper and lower limits.

3. Principle of mathematical induction. Geometric series. Bernoulli inequality. Factorial. Binomial coefficients. Newton's binomial theorem. $\lim_{n\to\infty} a^{\frac{1}{n}} = 1 (\forall a > 0)$. $x_n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ is convergent. Euler number.

4. Function. Domain and codomain. Real function. Operations with functions. Ways of defining a function: analytic, graphical, implicit, superposition, limits of sequences. Exponential function $a^x (a > 0, x \in \mathbb{R})$. Properties of exponential functions. Power function $x^\alpha (x > 0, \alpha \in \mathbb{R})$. Limit of a function through limits of sequences.

5. Properties of limits. Left and right limits. Infinite limits and limits at infinity. $\varepsilon - \delta$ definitions of the limit, left limit, right limit, infinite limit, limit at infinity. Cauchy criterion. Bounded function. Supremum and infimum of functions. Monotone increasing and monotone decreasing functions. Continuous, left-coutinuous, and right-continuous function at a point. Examples of continuous functions. Operations with continuous functions. 1st and 2nd Weierstrass theorems. Intermediate value theorem and its corollaries.

6. Strictly monotone increasing and decreasing functions. Inverse function. Logarithmic function and its properties. Continuity of the power function. Trigonometric functions and their continuity.

7. Examples of limits. $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (and corollaries). $\lim_{x\to 0} (1 + x)^{\frac{1}{x}} = e$ (and corollaries). Computation of limits using substitutions.

8. Derivative of a function. Differentiable function. Left and right derivatives. c', $(x^n)'$, $(\sin x)'$, $(\cos x)'$, $(a^x)'$. Continuity of differentiable function. Physical interpretation of derivative, examples. Geometric interpretation of derivative. Derivatives of the sum, product, ratio of functions. $(\tan x)'$, $(\cot x)'$.

9. Derivative of the inverse function. Derivatives of the inverse trigonometric functions. $(\log_a x)'$. Chain rule. Derivative of the power function. $\left(\frac{1}{2} \ln \left|\frac{x-a}{x+a}\right|\right)'$. $\left(\ln \left|x + \sqrt{x^2 + a}\right|\right)'$. Higher order derivatives. Formulas for $(a^x)^{(n)}$, $(\sin x)^{(n)}$, $(\cos x)^{(n)}$. Formulas for $(y_1 + y_2)^{(n)}$, $(y_1 y_2)^{(n)}$.

10. Sufficient condition for strict monotonicity of differentiable function. Local extrema (maximum and minimum). Point of strict increase or decrease of a function. Necessary condition for a local extremum (Fermat theorem). Sufficient condition for local maximum or minimum for twice differentiable functions. Generalization of the previous theorem to n-times differentiable functions - points of local maximum / minimum or local increase / decrease. Sufficient condition for local maximum / minimum for continuous function not differentiable at a point. Lagrange (mean value) theorem and its corollaries.

11. L'Hopital's rule. Various examples of limits with indeterminancies $\frac{\infty}{\infty}$, $\frac{0}{0}$, 0^0, $\infty - \infty$, including $\lim_{x\to+\infty} \frac{\ln x}{x^\alpha} = 0$ $(\alpha > 0)$, $\lim_{x\to+\infty} \frac{x^n}{a^x} = 0$ $(n \in \mathbb{N}, a > 1)$, $\lim_{x\to 0+} x^x = 1$.

12. Approximation of differentiable functions by polynomials. Taylor theorem. Taylor formula. Taylor polynomial. Maclaurin formula. Lagrange form of the remainder. Taylor formulas for $\sin x$, $\cos x$, e^x, e^{-x}, $\sinh x$, $\cosh x$, $\ln(1+x)$. Computation of limits using Taylor formula. $\lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x} = -\frac{e}{2}$.

13. Antiderivative of a function. Indefinite integral. Differential of a function $df(x) = f'(x)dx$. $\int f(x)dx \neq \int f(y)dy$. Linearity of antiderivatives. Table of integrals. Substitution rule, examples. $\int \frac{dx}{\sqrt{ax^2+bx+c}}$, $\int \frac{dx}{ax^2+bx+c}$. Integration by parts. $\int xe^x dx$. $\int \sqrt{a^2 - x^2}dx$. Trigonometric substitutions, $\int \frac{dx}{\cos^4 x}$. $\int e^{\alpha x} \cos \beta x dx$. Antiderivatives $\int \frac{e^x}{x^n}dx$, $\int \frac{\sin x}{x^n}dx$, $\int \frac{\cos x}{x^n}dx$, $\int e^{-x^2}dx$, elliptic integrals $\int \frac{dx}{\sqrt{1-k^2\sin^2 x}}$ and $\int \sqrt{1 - k^2 \sin^2 x} dx$ cannot be expressed as elementary functions.

14. Partition of an interval. Mesh of a partition. Riemann sum. Riemann integral. Riemann integrable function. Sufficient conditions for integrability: (a) boundedness, (b) continuity, (c) monotonicity. Basic properties of Riemann integral. $F(x) = \int_a^x f(t)dt$. F is continuous, and differentiable at x_0 if f is continuous at x_0. If f is continuous, then it has an antiderivative. Fundamental theorem of calculus. Substitution rule. Integration by parts.

15. Applications of integrals. Areas: disc, circular sector, regions under sine and hyperbola. Volumes of resolutions: ball, circular cone. Length of a curve: ellipse, astroid. Work of a force.

16. Improper integrals (of unbounded functions or on unbounded intervals). Convergent and divergent integrals. Properties. Fundamental theorem of calculus. $\int_0^1 \frac{dx}{x^\alpha}$ and $\int_1^\infty \frac{dx}{x^\alpha}$ $(\alpha > 0)$. $\int_0^1 (\ln x)^n dx$. Comparison criterion. $\int_0^1 \frac{dx}{\ln x}$, $\int_1^\infty \frac{\ln x}{x^2}dx$. Absolute convergence.

17. Series. Converent series. The sum of a series. Basic properties. Necessary condition for convergence. Series with positive elements. Integral criterion for convergence. $\sum_{k=1}^\infty \frac{1}{k^\alpha}$ $(\alpha > 0)$. Comparison criterion. Study of convergence (or divergence) of series using Taylor formula. The ratio test. $\sum_{k=1}^\infty \frac{k}{e^k}$. Root test. $\sum_{k=1}^\infty \frac{1}{k^k}$. Series with arbitrary elements. Dirichlet test. $\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k}$, $\sum_{k=1}^\infty \frac{\cos k\alpha}{\sqrt{k}}$. Absolutely convergent series. Any permutation of elements of absolutely convergent series does

not change its sum. Sum and product of absolutely convergent series. Riemann theorem (about the sum of a series convergent but not absolutely).

18. Series of functions. Convergence and absolute convergence. Uniform convergence. Weierstrass M-test. $\sum_{k=1}^{\infty} \frac{x^k}{k!}$. Continuity of the sum. Integration and differentiation of series of functions. Explicit computation of $\sum_{k=1}^{\infty} kq^k$ and $\sum_{k=1}^{\infty} \frac{q^k}{k}$ $(q \in (0,1))$. Power series. Radius of convergence. Cauchy-Hadamard formula. Real-analytic functions. Properties of analytic functions. Analytic functions are infinitely differentiable. Taylor series. Uniform boundedness of all derivatives implies that the function equals to its Taylor series. Taylor series for e^x, $\sin x$, $\cos x$, $\ln(1+x)$.

19. Complex numbers. Imaginary unit. Algebraic form of complex number. Real and imaginary parts. Complex conjugate. Modulus. Properties of addition and multiplication. Subtration and division of complex numbers. Geometric interpretation. Triangle inequality. Argument of $z \neq 0$. Trigonometric form of complex numbers. Euler formula, definition of $e^{i\varphi}$. De Moivre formula. Computations with complex numbers, $\sum_{k=0}^{n} \cos(x+k\alpha)$, $\sum_{k=0}^{n} \sin(x+k\alpha)$. Exponential form of complex numbers. Complex roots. Fundamental theorem of algebra. Limits of sequences of complex numbers, complex valued functions. Complex exponential. Series of complex valued functions. Cauchy-Hadamard formula. Definition of e^z, $\sin z$, $\cos z$ using Taylor series.

20. Vector. Collinear and coplanar vectors. Operations with vectors and their properties. Linear combination. Basis in $\mathbb{R}^2$ and $\mathbb{R}^3$. Coordinates of vectors. Linearly independent vectors. Criteria for linear dependence of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$. Affine coordinate system. Cartesian coordinate system. Polar, spherical, and cylindrical coordinates. Scalar product and its properties. Scalar product of vectors through their coordinates in an orthonormal basis. Orientations of triples of non-coplanar vectors in space. Vector product and its properties. Scalar triple product. Volume of parallelepiped spanned by 3 vectors. Vector and scalar triple products of vectors through their coordinates in an orthonormal basis with right-hand orientation, their representations as determinants. General equations of lines in $\mathbb{R}^2$ and planes in $\mathbb{R}^3$, meaning of coefficients in case the coordinate system is Cartesian.

21. Systems of linear equations. Consistent and inconsistent systems. Equivalent systems. Elementary operations with systems. Matrix and augmented matrix of the system. Elementary row operations with matrices. Row echelon form and reduced row echelon form. Gaussian elimination. Homogeneous system. Closedness of the solutions set to a homogeneous system under addition and multiplication by scalars.

22. Field. Examples. Field with only 2 elements. Vector space over a field. Examples. Linear combination of vectors. Non-trivial linear combination. Linear independence and linear dependence, properties. Complete system of vectors. Basis of vector space. Finite and infinite dimensional vector spaces. Examples of bases. Coordinates of vectors in a basis. Coordinates of sums of vectors and a vector multiplied by a scalar. Every complete system contains a basis and every linearly independent

system can be enhanced to a basis. All bases of finite dimensional vector spaces have the same number of vectors. Dimension of vector space.

23. Injective, surjective, and bijective maps. Isomorphism of vector spaces, and its properties. Every n-dimensional vector space over a field F is isomorphic to F^n. Vector subspaces. Examples. Span of set of vectors. Span as the smallest vector subspace containing given vectors. Examples. Rank of set of vectors. Rank as the maximal number of linearly independent vectors. If U is a vector subspace of V and $\dim(U) = \dim(V)$, then $U = V$. Intersection and sum of vector subspaces. Dimension formula. Direct sum of vector subspaces.

24. Linear map. Linear operation. Linear functional. Examples. Kernel and image of linear map. Kernel and image are vector subspaces. Kernel is zero space if and only if the map is injective. Dimension formula. Rank of linear map. Inverse to linear map. Inverse exist if and only if the map is bijective. Properties of the inverse map. Sum, multiplication by scalars, and composition of linear maps. Vector space of linear maps $\mathcal{L}(U, V)$. Matrix of linear map. Example: change of basis. Set of matrices $M_{m,n}(F)$. Bijection of the sets $\mathcal{L}(U, V)$ and $M_{m,n}(F)$ for fixed bases of n-dimensional U and m-dimensional V.

25. Matrix of sum of linear maps and sum of matrices. Matrix of map multiplied by scalar and multiplication of matrices by scalars. $M_{m,n}(F)$ is a vector space. Given bases of n-dimensional U and m-dimensional V, $\mathcal{L}(U, V)$ and $M_{m,n}(F)$ are isomorphic. Matrix of composition of linear maps and product of matrices. Product of matrices is an associative operation but generally non-commutative. Row and column vectors. System of linear equations in matrix notation. Identity matrix. Matrix of inverse map and inverse to a matrix. Map is invertible if and only if its matrix (in some bases) is invertible.

26. Rank of matrix as maximal number of linearly independent rows and maximal number of linearly independent columns. Rank is preserved under elementary row and column transformations. Rank of linear map equals to rank of its matrix (in some bases). $n \times n$ matrix is invertible if and only if its rank is n. Computation of inverse matrix using Gaussian elimination. Two matrices are both invertible if and only if their product is invertible. $\operatorname{rank}(AB) \leq \min(\operatorname{rank}(A), \operatorname{rank}(B))$. Example of $\operatorname{rank}(AB) \neq \operatorname{rank}(BA)$. Dimension of solution set to homogeneous system of equation as the number of unknowns minus the rank of the system matrix.

27. Determinant of matrix. Examples: determinants of 2×2 and 3×3 matrices, determinant of the identity matrix. Expansion of determinant along any row or any column. Properties of determinants. The following claims are equivalent for $A \in M_{n,n}(F)$: (a) determinant of A is non-zero, (b) all the rows of A are linearly independent, (c) all the columns of A are linearly independent, (d) rank of A is n, (e) A is invertible, (f) system of linear equations with matrix A has unique solution, (g) A can be transformed by elementary row operations to the identity matrix. Determinant as an alternating multilinear function of columns properly normalized.

Leibniz formula for the determinant (as the sum over permutations of $\{1, \dots, n\}$). Examples: 2×2 and 3×3 matrices.

28. Choice of bases of vector spaces U and V such that given linear map between them of rank r has matrix $\begin{pmatrix} I_r & 0_{r,n-r} \\ 0_{m-r,m} & 0_{m-r,n-r} \end{pmatrix}$. Any matrix of rank r can be written as the product $P \cdot \begin{pmatrix} I_r & 0_{r,n-r} \\ 0_{m-r,m} & 0_{m-r,n-r} \end{pmatrix} \cdot Q$, for some invertible P and Q. Invariant subspace. Examples. Restriction of linear map to invariant subspace and its matrix. Eigenvectors and eigenvalues of linear operators. Spectrum of linear operator.

29. Characteristic polynomial. Independence of characteristic polynomial of basis. Similar matrices. Eigenvalues as roots of characteristic polynomial. Examples. Eigenvectors corresponding to different eigenvalues are linearly independent. Simple spectrum. Matrix of linear operator with simple spectrum is diagonalizable. Algebraic and geometric multiplicities of eigenvalues. Eigenspace. Geometric multiplicity is at most algebraic multiplicity, they are equal if and only if the matrix of operator is diagonalizable.