**RETAKE SOLUTIONS, 04 April 2016, 13:00 – 15:00**

1. (4 points) For which $n \in \mathbb{N}$ the following inequality holds?

$$2^n > 2n + 1$$

*Answer:* $n \geq 3$.

*Solution.* Direct calculation gives that the inequality is not true for $n = 1$ and $n = 2$, but true for $n = 3$. ($2^3 = 8 > 7 = 2 \cdot 3 + 1$) We prove by induction on $n$ that the inequality is true for all $n \geq 3$. The base of induction is verified, it remains to check the induction step. Assume that the inequality is true for $n = k$, where $k$ is some integer $\geq 3$, and show that the inequality is also true for $n = k + 1$. By the induction hypothesis,

$$2^{k+1} = 2\,2^k \geq 2\,(2k+1) = 4k + 2 = 2(k+1) + 1 + (2k-1) \geq 2(k+1) + 1.$$

Thus, the inequality is true for $n = k + 1$. We conclude that the inequality holds if and only if $n \geq 3$. □

2. (4 points) Compute the limit

$$\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}.$$

*Answer:* $e^{-\frac{1}{2}}$.

*Solution.* Note that

$$\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}} = e^{\lim_{x\to 0} \frac{\ln \cos x}{x^2}}.$$

Using the facts that $\lim_{y\to 0} \frac{\cos y - 1}{y^2} = -\frac{1}{2}$ and $\lim_{y\to 0} \frac{\ln(1+y)}{y} = 1$, we obtain that

$$\begin{aligned}\lim_{x\to 0} \frac{\ln \cos x}{x^2} &= \lim_{x\to 0} \frac{\ln(1 + (\cos x - 1))}{\cos x - 1} \cdot \frac{\cos x - 1}{x^2} \\ &= \lim_{x\to 0} \frac{\ln(1 + (\cos x - 1))}{\cos x - 1} \cdot \lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{y\to 0} \frac{\ln(1+y)}{y} \cdot \lim_{x\to 0} \frac{\cos x - 1}{x^2} \\ &= 1 \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2}.\end{aligned}$$

Another way to show that $\lim_{x\to 0} \frac{\ln \cos x}{x^2} = -\frac{1}{2}$ is by using l'Hospital's rule. □

3. (4 points) Is the following function continuous? Is it differentiable?

$$f(x) = \begin{cases} \sin x \cdot \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

*Answer: continuous on* $\mathbb{R}$, *differentiable on* $\mathbb{R} \setminus \{0\}$.

*Solution.* First of all, both $\sin x$ and $\sin \frac{1}{x}$ are differentiable for all $x \neq 0$. Thus, $f$ is differentiable for all $x \neq 0$. Since differentiability implies continuity, $f$ is continuous at every $x \neq 0$. It remains to examine the point $x = 0$.

Since $|\sin x \sin \frac{1}{x}| \leq |\sin x|$ and $\lim\limits_{x\to 0} |\sin x| = 0$, we conclude that $\lim\limits_{x\to 0} f(x) = 0 = f(0)$. Thus, $f$ is continuous at 0.

Next, $f$ is differentiable at 0 if the limit $\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0}$ exists. We compute

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\sin x \sin \frac{1}{x}}{x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \sin \frac{1}{x}.$$

Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \sin \frac{1}{x}$ does not exist (for instance, if $x_n = \frac{1}{2\pi n}$ and $y_n = \frac{1}{\frac{\pi}{2}+2\pi n}$, then $\lim_{n\to\infty} \sin \frac{1}{x_n} = 0$ and $\lim_{n\to\infty} \sin \frac{1}{y_n} = 1$), we conclude that the limit $\lim_{x\to 0} \frac{\sin x}{x} \cdot \sin \frac{1}{x}$ does not exist. Thus, $f$ is not differentiable at 0. □

4. (4 points) Prove that the sequence $x_n = ne^{-n}$, $n \in \mathbb{N}$, is monotone decreasing.

*Solution.* Consider the function $f(x) = xe^{-x}$. This function is monotone decreasing for $x \geq 1$. Indeed,

$$f'(x) = e^{-x} - xe^{-x} = (1 - x)e^{-x}.$$

Thus, $f'(x) \leq 0$ if and only if $x \geq 1$. Since $x_n = f(n)$, the sequence $x_n$ is monotone decreasing. □

5. (4 points) Find the global minimum of the function $f(x) = \frac{1}{2}|x| + \sin x$, $x \in \mathbb{R}$.

*Answer:* $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$.

*Solution.* First note that $f$ is differentiable everywhere except $x = 0$, and $f(0) = 0$.

Next, $f(x) \geq \frac{1}{2}|x| - 1 > 0$ if $|x| > 2$. Thus, the minimum of $f$ can only be attained at some $x \in [-2, 2]$. Since $f$ is continuous, the minimum is indeed attained at some $x \in [-2, 2]$. Moreover, if $x \in (0, 2]$, then $\sin x > 0$ and $f(x) = \frac{1}{2}|x| + \sin x > 0$. Thus, the minimum is attained at some $x \in [-2, 0]$. In particular, it coincides with the minimum of the function $g(x) = -\frac{1}{2}x + \sin x$ on the interval $[-2, 0]$.

The global minimum of $g$ on $[-2, 0]$ is attained either at $-2$, or at 0, or at some $x \in [-2, 0]$ where $g'(x) = 0$. Note that $g(-2) \geq 0$ and $g(0) = 0$. Also note that $g'(x) = -\frac{1}{2} + \cos x = 0$ if and only if $\cos x = \frac{1}{2}$ if and only if $x = \pm\frac{\pi}{3} + 2\pi k$, where $k \in \mathbb{Z}$. The only $x$ that is in $[-2, 0]$ is $x = -\frac{\pi}{3}$, and

$$g(-\frac{\pi}{3}) = \frac{\pi}{6} + \sin(-\frac{\pi}{3}) = \frac{\pi}{6} - \frac{\sqrt{3}}{2}.$$

Since $\frac{\pi}{6} - \frac{\sqrt{3}}{2} \leq \frac{4}{6} - \frac{\sqrt{3}}{2} < 0$, we conclude that the global minimum of $f$ is attained at $x = -\frac{\pi}{3}$ and equals $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$.

□

6. (3 points) Compute the second derivative of the function $f(x) = \sin(\sin x)$.
   *Answer:* $-\sin(\sin x)\cos^2 x - \cos(\sin x)\sin x$.

   *Solution.* We compute $f'(x) = \cos(\sin x)\cos x$,
   $f''(x) = (\cos(\sin x))'\cos x + \cos(\sin x)(\cos x)' = -\sin(\sin x)\cos^2 x - \cos(\sin x)\sin x$.
   □

7. (4 points) Compute the integral

$$\int_1^2 x\ln x dx.$$

   *Answer:* $2\ln 2 - \frac{3}{4}$.

   *Solution.* By the integration by parts formula,

$$\int_1^2 x\ln x dx = \frac{x^2}{2}\ln x\Big|_1^2 - \int_1^2 \frac{x^2}{2}\frac{dx}{x} = 2\ln 2 - \frac{1}{2}\frac{x^2}{2}\Big|_1^2 = 2\ln 2 - \frac{3}{4}.$$

   □

8. (4 points) Does the following improper integral converge?

$$\int_0^\infty \frac{\ln(1+x)}{x\sqrt{x}}dx.$$

   *Answer:* *yes.*

   *Solution.* First note that $\frac{\ln(1+x)}{x\sqrt{x}} > 0$ for all $x > 0$. We consider separately improper integrals $I_1 = \int_0^1 \frac{\ln(1+x)}{x\sqrt{x}}dx$ and $I_2 = \int_1^\infty \frac{\ln(1+x)}{x\sqrt{x}}dx$.
   For the first one, since $\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$, the integral $I_1$ converges if and only if the integral with $\frac{\ln(1+x)}{x}$ replaced by 1 converges, i.e., $\int_0^1 \frac{1}{\sqrt{x}}dx$. Since the latter does converge, the integral $I_1$ also converges.
   For the second one, since $\ln(1+x) \le x^{\frac{1}{4}}$ for all large $x$, the integral $I_2$ converges if the integral $\int_1^\infty \frac{x^{\frac{1}{4}}}{x\sqrt{x}}dx = \int_1^\infty \frac{1}{x^{\frac{5}{4}}}dx$ converges. Since the latter does converge, the integral $I_2$ also converges.
   Since both improper integrals $I_1$ and $I_2$ converge, the improper integral $\int_0^\infty \frac{\ln(1+x)}{x\sqrt{x}}dx$ also converges. □

9. (4 points) Does the following series converge?

$$\sum_{n=1}^\infty \sin\frac{1}{\sqrt{n}}.$$

   *Answer:* *no.*

*Solution.* Since $\lim_{n\to\infty} \frac{\sin\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = 1$, the series $\sum_{n=1}^{\infty} \sin\frac{1}{\sqrt{n}}$ converges if and only if the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges. The latter diverges, thus the given series also diverges. □

10. (3 points) Write the complex number $\frac{5}{2-i}$ in the algebraic form.

*Answer:* $2+i$.

*Solution.*

$$\frac{5}{2-i} = \frac{5(2+i)}{(2-i)(2+i)} = \frac{5(2+i)}{4-i^2} = 2+i.$$

□

11. (3 points) Give the definition of the radius of convergence of the complex series $\sum_{n=0}^{\infty} a_n\, z^n$, where $a_n \in \mathbb{C}$, $z \in \mathbb{C}$.

12. (3 points) Give the definition of linear independence of vectors $v_1, \ldots, v_n$ in the vector space $V$ over a field $F$.

13. (3 points) Are the vector spaces $\mathbb{R}$ and $\mathbb{R}^2$ isomorphic? Justify your answer.

*Answer: no.*

*Solution.* The dimension of $\mathbb{R}$ is 1, and the dimension of $\mathbb{R}^2$ is 2, thus they are not isomorphic. □