RETAKE, 10.04.2015, 15:15 – 17:15
(Solutions)

1. (4 points) Prove that $7^n - 1$ is divisible by 3 for all integers $n \geq 1$.

   Solution. We use induction on n. Base of induction: for $n = 1$, $7^1 - 1 = 6$ is divisible by 3. Induction step: assume that for some $n \in \mathbb{N}$, $7^n - 1$ is divisible by 3, and prove that $7^{n+1} - 1$ is also divisible by 3. Indeed, $7^{n+1} - 1 = 7(7^n - 1) + 6$, which is divisible by 3 by the induction assumption. □

2. (4 points) Compute the limit $\lim_{x\to 0} \frac{\frac{\sin x}{x} - 1}{x^2}$.

   Solution. First solution: By Taylor's theorem, $\sin x = x - \frac{x^3}{6} + o(x^3)$, as $x \to 0$. Thus,

$$\lim_{x\to 0} \frac{\frac{\sin x}{x} - 1}{x^2} = \lim_{x\to 0} \frac{\frac{x - \frac{x^3}{6} + o(x^3)}{x} - 1}{x^2} = -\frac{1}{6}.$$

   Second solution: Rewrite $\frac{\frac{\sin x}{x} - 1}{x^2} = \lim_{x\to 0} \frac{\sin x - x}{x^3}$. Since the nominator and denominator both tend to 0 as $x \to 0$, by the l'Hopital rule,

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{\cos x - 1}{3x^2} = -\frac{1}{6}.$$

   □

3. (4 points) For which values of $a \in \mathbb{R}$ the following function is continuous at 0? For which a it is differentiable at 0?

$$f(x) = \begin{cases} \frac{\sin x}{x^a} & \text{for } x \neq 0, \\ 1 & \text{for } x = 0. \end{cases}$$

   Solution. Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and

$$\lim_{x\to 0} x^b = \begin{cases} 0 & \text{if } b > 0, \\ 1 & \text{if } b = 0, \\ \infty & \text{if } b < 0, \end{cases}$$

   we obtain that

$$\lim_{x\to 0} \frac{\sin x}{x^a} = \lim_{x\to 0} \frac{\sin x}{x} \cdot x^{1-a} = \begin{cases} 0 & \text{if } a < 1, \\ 1 & \text{if } a = 1, \\ \infty & \text{if } a > 1, \end{cases}$$

   Thus, f is continuous at 0 only if $a = 1$.

Since differentiability at 0 implies continuity at 0, we immediately conclude that f is not differentiable at 0 for any $a \neq 1$. It remains to check the case $a = 1$. By the definition, we need to check if the following limit exists:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{\frac{\sin x}{x} - 1}{x} = \lim_{x\to 0} \frac{\frac{\sin x}{x} - 1}{x^2} \cdot x = \lim_{x\to 0} \frac{\frac{\sin x}{x} - 1}{x^2} \cdot \lim_{x\to 0} x = -\frac{1}{6} \cdot 0 = 0.$$

(In the last step we used the result of problem 2.) Thus, for $a = 1$, the derivative of f at 0 exists (and equals 0).

Answer: The function is continuous at 0 only for $a = 1$, and the function is differentiable at 0 only for $a = 1$. □

4. (4 points) Find the global maximum and minimum of the function $f(x) = \frac{\ln x}{x}$ on the interval $[1, 3]$.

Solution. First of all, by the extreme value theorem, since f is continuous on the closed bounded interval $[1, 3]$, it attains its maximum and minimum on $[1, 3]$.

We compute

$$f'(x) = \frac{(\ln x)' x - \ln x x'}{x^2} = \frac{1 - \ln x}{x^2} \begin{cases} > 0 & \text{if } x \in (0, e), \\ = 0 & \text{if } x = e, \\ < 0 & \text{if } x > e. \end{cases}$$

Note that $e \in [1, 3]$. Thus, the function is increasing on $[1, e]$ and decreasing on $[e, 3]$. The maximum of f on $[1, 3]$ equals $f(e) = \frac{1}{e}$, and the minimum is $\min(f(1), f(3)) = \min(0, \frac{\ln 3}{3}) = 0$. □

5. (4 points) For $x \geq 1$, find the area bounded by the curve $y = xe^{-x^2}$ and the x-axis.

Solution. The area equals $\int_1^\infty xe^{-x^2} dx$. Using the substitution $y = x^2$ and $dy = 2xdx$ (with limits of integration remaining 1 and ∞), we compute

$$\int_1^\infty xe^{-x^2} dx = \int_1^\infty e^{-y} dy = -e^{-y}|_1^\infty = -\left(\lim_{y\to\infty} e^{-y} - e^{-1}\right) = e^{-1}.$$

□

6. (4 points) Compute the indefinite integral $\int x \sin x dx$.

Solution. Using integration by parts ($u = x$, $dv = \sin x dx$; $du = dx$, $v = -\cos x$),

$$\int x \sin x dx = x \cdot (-\cos x) - \int (-\cos x) dx = -x \cos x + \sin x + C.$$

□

7. (4 points) For which positive real numbers a the improper integral $\int_1^2 \frac{dx}{(\ln x)^a}$ converges?

Solution. Since the denominator equals 0 if $x = 1$, and since $\ln x = \ln(1+(x-1)) = (x-1) + o(x-1)$, as $x \to 1$, the improper integral $\int_1^2 \frac{dx}{(\ln x)^a}$ converges if and only if the improper integral $\int_1^2 \frac{dx}{(x-1)^a}$ converges.

Using the substitution $y = x-1$ and $dy = dx$ (and changing the limits of integration from 1 and 2 to 0 and 1), we rewrite the last integral as $\int_0^1 \frac{dy}{y^a}$. We know that this improper integral converges if and only if $a \in (0,1)$. Thus, the original improper integral also converges if and only if $a \in (0,1)$. □

8. (4 points) For which $a \in \mathbb{R}$ the series $\sum_{n=2}^{+\infty} \frac{(-1)^n}{(\ln n)^a}$ converges?

   Solution. The series diverges for $a \le 0$, since the necessary condition for convergence is violated: $\lim_{n\to\infty} \frac{(-1)^n}{(\ln n)^a} \neq 0$. For $a > 0$, we apply the Dirichlet test. The sequence $a_n = \frac{1}{(\ln n)^a}$ is monotone decreasing to 0, and $b_n = (-1)^n$ satisfies $|\sum_{n=2}^M b_n| \le 1$ for all M. Thus, by the Dirichlet test, the series $\sum_{n=2}^\infty a_n b_n$ converges.

   Answer: the series converges for $a > 0$ and diverges for $a \le 0$. □

9. (4 points) Find all the solutions to the equation $z^3 = 1 + i$ in the form $z = re^{i\theta}$, where $r > 0$ and $\theta \in [0, 2\pi)$.

   Solution. Since $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}e^{i\frac{\pi}{4}}$, the problem is equivalent to finding $r > 0$ and $\theta \in [0, 2\pi)$ such that $r^3 = \sqrt{2}$ and $3\theta \in \{\frac{\pi}{4} + 2\pi n \;:\; n \in \mathbb{Z}\}$.

   Answer: $r = \sqrt[6]{2}$, $\theta \in \{\frac{\pi}{12}, \frac{9\pi}{12}, \frac{17\pi}{12}\}$. □

10. (4 points) Let $M = \{(1,0,0,0), (1,1,1,2), (1,2,2,1), (3,0,0,1)\}$ be a set of vectors in $\mathbb{R}^4$. Find the rank of M.

    Solution. The rank of M is the maximal number of linearly independent vectors in M, which is equal to the rank of the matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 3 & 0 & 0 & 1 \end{pmatrix}$. To compute the rank of this matrix we use Gaussian elimination. We subtract the first row from the second and the third and subtract the first row multiplied by 3 from the fourth, to arrive at equivalent matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. We then add the last row multiplied by 3 to the third row to get the matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Finally, we subtract the second row multiplied by 2 from the third row to get $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. The rank of this matrix is 3. Since the above operations preserve the rank, the rank of the original matrix (and of the set M) is also 3. $\square$