Exercises 7.1, Mathematics 1 (12-PHY-BIPMA1)

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1. Let $a \in \mathbb{R}$ and $x \geq -1$, $x \neq 0$. Use Lagrange's theorem to prove the following inequalities:

   (a) if $0 < a < 1$ then $(1+x)^a < 1 + ax$,

   (b) if $a < 0$ or $a > 1$ then $(1+x)^a > 1 + ax$.

   Hint: Apply Lagrange's theorem to function $f(z) = (1+z)^a$ on the interval $[0, x]$ (if $x > 0$) or $[x, 0]$ (if $x < 0$).

2. Use l'Hopital's rule to compute the limits:

   (a) $\lim_{x\to a} \frac{a^x - x^a}{x-a}$, for $a > 0$,

   (b) $\lim_{x\to 0+} x^\varepsilon \ln x$, for $\varepsilon > 0$,

   (c) $\lim_{x\to 1} x^{\frac{1}{1-x}}$,

   (d) $\lim_{x\to 0} \left(\cot x - \frac{1}{x}\right)$,

   (e) $\lim_{x\to e} \frac{\ln(\ln x)}{\sin(x-e)}$,

   (f) $\lim_{x\to +\infty} \sqrt{x}\left(e^{-\frac{1}{x}} - 1\right)$.

3. Consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

   Compute $f^{(n)}(0)$ for all $n \in \mathbb{N}$.