Exercises 2.1, Mathematics 1 (12-PHY-BIPMA1)
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1. Let $\lim_{n\to\infty} a_n = a$. Let (b_n) be a subsequence of (a_n). Prove that $\lim_{n\to\infty} b_n = a$.

2. Prove convergence of the following sequences using Cauchy theorem:

   (a) $a_n = \frac{1}{n}$,

   (b) $a_n = \frac{1}{2^n}$,

   (c) $a_n = \frac{\sin n}{n}$.

3. Indentify all cluster points of the following sequences and compute their $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$:

   (a) $\frac{1}{n}$,

   (b) $(-1)^n$.

4. Let $a > 0$ and $x_1 > 0$. For $n \geq 2$, let $x_n = \frac{1}{2}\left(x_{n-1} + \frac{a}{x_{n-1}}\right)$. Prove that (x_n) is a convergent sequence and identify the limit. What can you say about the speed of convergence of x_n to its limit?

   (Hint:

   - show that $x_n \geq \sqrt{a}$ for all $n \geq 2$,
   - show that $x_{n+1} \leq x_n$ for all $n \geq 2$,
   - pass to the limit on the both sides of the recursive formula for x_n and identify the limit.

   As for the speed of convergence, prove that $0 \leq x_n - x_{n+1} \leq \frac{1}{2^{n-2}}(x_2 - x_3)$ and deduce from this an upper bound on $x_n - x_{n+k}$ for any k, then pass to the limit as $k \to \infty$.)