Exercises 14.1, Mathematics 1 (12-PHY-BIPMA1)
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1. Let $F$ be a field. Prove that
   (a) The multiplicative identily 1 is unique in $F$, i.e., if for some $\alpha \in F$, $\alpha \cdot \beta = \beta$ for all $\beta \in F$, then $\alpha = 1$.
   (b) For any $\alpha \in F$, its additive inverse $-\alpha \in F$ is unique.
   (c) For any $\alpha \in F$, its multiplicative inverse $\alpha^{-1} \in F$ is unique.

2. Let $V$ be a vector space over a field $F$. Prove that
   (a) For all $\alpha \in F$, $\alpha \cdot \overrightarrow{0} = \overrightarrow{0}$.
   (b) For all $u \in V$, $0 \cdot u = \overrightarrow{0}$.
   (c) For all $u \in V$, $(-1) \cdot u = -u$.

3. Let $V$ be a vector space over a field $F$. Let $u_1, \ldots, u_n$ be linearly independent, and $v_1, \ldots, v_m$ complete system of vectors. Prove that $n \leq m$.

4. Let $T$ be an isomorphism between vector spaces $V_1$ and $V_2$. Prove that
   (a) $T(\overrightarrow{0_{V_1}}) = \overrightarrow{0_{V_2}}$, i.e., the zero vector of $V_1$ is mapped to the zero vector of $V_2$.
   (b) For all $u \in V_1$, $T(-u) = -T(u)$, i.e., additive inverses are mapped to additive inverses.
   (c) If $u_1, \ldots, u_n$ are linearly independent in $V_1$, then $T(u_1), \ldots, T(u_n)$ are linearly independent in $V_2$.

5. Let $V_1, V_2, V_3$ be vector spaces over a field $F$. Prove that
   (a) $V_1 \cong V_1$,
   (b) if $V_1 \cong V_2$, then $V_2 \cong V_1$,
   (c) if $V_1 \cong V_2$ and $V_2 \cong V_3$, then $V_1 \cong V_3$.

   Properties (a)-(c) state that $\cong$ is an *equivalence relation* on vector spaces.

6. Let $U_1$ and $U_2$ be vector subspaces of a vector space $V$.
   (a) Prove that $U_1 \cap U_2$ and $U_1 + U_2$ are vector subspaces of $V$.
   (b) If $U_1$ is a vector subspace of $U_2$, prove that $U_1 \cap U_2 = U_1$ and $U_1 + U_2 = U_2$.