EXAM SOLUTIONS, 15 February 2016, 10:00 – 12:00

1. (4 points) Prove that for all natural numbers $n \geq 2$,

$$\frac{1}{n+1} + \ldots + \frac{1}{2n} \geq \frac{7}{12}.$$

Solution. By induction on n. Base of induction: For $n = 2$, $\frac{1}{3} + \frac{1}{4} = \frac{7}{12} \geq \frac{7}{12}$.

Induction step: Assume that the inequality holds for $n = k$ and consider $n = k + 1$. Then

$$\begin{aligned}\frac{1}{n+1} + \ldots + \frac{1}{2n} &= \frac{1}{(k+1)+1} + \ldots + \frac{1}{2(k+1)} = \frac{1}{k+2} + \ldots + \frac{1}{2k+2} \\ &= \left(\frac{1}{k+1} + \ldots + \frac{1}{2k}\right) + \left(\frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1}\right) \\ &\geq \frac{7}{12} + \left(\frac{1}{2k+1} - \frac{1}{2k+2}\right) \geq \frac{7}{12},\end{aligned}$$

where in the first inequality we used the induction hypothesis.

We have shown that the inequality holds for $n = 2$ and once it holds for $n = k$ it also holds for $n = k + 1$. By the principle of induction, the inequality holds for all $n \geq 2$. □

2. (3 points) For a sequence x_n of real numbers, give the definition of the upper limit $\limsup_{n\to\infty} x_n$ (or, in other notation, $\overline{\lim_{n\to\infty}} x_n$).

3. (4 points) For which $a \in \mathbb{R}$ the following function is continuous on $\mathbb{R}$?

$$f(x) = \begin{cases} \frac{\sin 3x}{x} & \text{for } x \neq 0 \\ a & \text{for } x = 0. \end{cases}$$

Answer: $a = 3$.

Solution. First notice that $\sin 3x$ and x are continuous functions on $\mathbb{R}$. Therefore, $f(x)$ is continuous at any $x \neq 0$ as the ratio of continuous functions for any choice of $a \in \mathbb{R}$.

It remains to examine the point $x = 0$. $f(x)$ is continuous at 0 if $\lim_{x\to 0} f(x) = f(0)$. Since $f(0) = a$ and

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin 3x}{x} = 3 \lim_{x\to 0} \frac{\sin 3x}{3x} = 3 \lim_{y\to 0} \frac{\sin y}{y} = 3,$$

f is continous at 0 if and only if $a = 3$. □

4. (4 points) Compute the limit
$$\lim_{x\to 0}\frac{e^x-1}{\sin 2x}\,.$$

Answer: $\frac{1}{2}$.

Solution.
$$\lim_{x\to 0}\frac{e^x-1}{\sin 2x}=\lim_{x\to 0}\left(\frac{e^x-1}{x}\cdot\frac{2x}{\sin 2x}\cdot\frac{1}{2}\right)=\left(\lim_{x\to 0}\frac{e^x-1}{x}\right)\cdot\left(\lim_{x\to 0}\frac{2x}{\sin 2x}\right)\cdot\frac{1}{2}=1\cdot 1\cdot\frac{1}{2}=\frac{1}{2}.$$
□

5. (4 points) Is the function $f(x)=|x|\sin x$ differentiable on $\mathbb{R}$?

Answer: yes.

Solution. For $x\neq 0$, both $|x|$ and $\sin x$ are differentiable functions. Thus, $f(x)$ is differentiable at any $x\neq 0$ as a product of differentiable functions.

f is differentiable at 0 if the following limit exists:
$$\lim_{x\to 0}\frac{f(x)-f(0)}{x}=\lim_{x\to 0}\frac{|x|\sin x}{x}.$$

Since $0\le|\frac{|x|\sin x}{x}|=|\sin x|$ and $\lim_{x\to 0}|\sin x|=0$, the above limit exists and equals to 0. Thus, f is differentiable at 0 and $f'(0)=0$. □

6. (3 points) Let $f(x)=e^{\sin x}$. Compute $f'(x)$ and $f''(x)$.

Answer: $f'(x)=e^{\sin x}\cos x$, $f''(x)=e^{\sin x}(\cos^2 x-\sin x)$.

7. (4 points) Prove that $\frac{\ln 17}{\ln 19}>\frac{17}{19}$.

Solution. Since $\ln 19>0$, we may rewrite the inequality as $\frac{\ln 17}{17}>\frac{\ln 19}{19}$.

Consider the function $f(x)=\frac{\ln x}{x}$. It suffices to prove that f is strictly monotone decreasing on the interval $[17,19]$. For this it suffices to show that $f'(x)<0$ for all $x\in[17,19]$. We have
$$f'(x)=\frac{(\ln x)'x-(x)'\ln x}{x^2}=\frac{1-\ln x}{x^2},$$
which is negative for all $x>e$. Since $e<3$, $f'(x)<0$ for all $x\in[17,19]$. □

8. Compute the integrals
$$\text{(a) (4 points)}\quad \int_e^{e^2}\frac{\ln(\ln x)}{x}dx \qquad \text{(b) (4 points)}\quad \int_0^\infty x^3e^{-x^2}dx\,.$$

Answer: (a) $2\ln 2-1$, *(b)* $\frac{1}{2}$.

Solution. (a) Make the substitution $y = \ln x$, $dy = \frac{1}{x}dx$ and change the limits of integration to $\ln e = 1$ and $\ln e^2 = 2$. Then compute using integration by parts

$$\int_e^{e^2} \frac{\ln(\ln x)}{x} dx = \int_1^2 \ln y dy = (y \ln y)\Big|_1^2 - \int_1^2 dy = 2\ln 2 - 1.$$

(b) This is an improper integral. First make the substitution $y = x^2$, $dy = 2xdx$ and change the limits of integration to $0^2 = 0$ and $(\lim_{x\to\infty} x)^2 = \infty$. Then compute using integration by parts

$$\int_0^\infty x^3 e^{-x^2} dx = \frac{1}{2}\int_0^\infty y e^{-y} dy = \frac{1}{2}\left((-ye^{-y})\Big|_0^\infty - \int_0^\infty (-e^{-y}) dy \right)$$
$$= \frac{1}{2}\left(0 + (-e^{-y})\Big|_0^\infty\right) = \frac{1}{2}.$$

□

9. (4 points) For which $a > 0$ the following series converges?

$$\sum_{n=1}^\infty \left(\sin\frac{1}{n}\right)^a .$$

Answer: $a > 1$.

Solution. Note that $\sin\frac{1}{n} > 0$ for all $n \geq 1$, thus also $\left(\sin\frac{1}{n}\right)^a > 0$. We can use the comparison criterion. Since $\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$, the series $\sum_{n=1}^\infty \left(\sin\frac{1}{n}\right)^a$ converges if and only if the series $\sum_{n=1}^\infty \left(\frac{1}{n}\right)^a$ converges. The latter is known to converge if and only if $a > 1$. □

10. (4 points) Prove that $\left(\sqrt{3}+i\right)^{2016}$ is a real number.

Solution. Rewrite

$$\left(\sqrt{3}+i\right)^{2016} = 2^{2016}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)^{2016} = 2^{2016}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{2016} .$$

By de Moivre's formula, $(\cos\varphi + i\sin\varphi)^n = \cos(n\varphi) + i\sin(n\varphi)$, the above expression equals $2^{2016}\left(\cos\frac{2016\pi}{6} + i\sin\frac{2016\pi}{6}\right)$. Since 2016 is divisible by 6, $\sin\frac{2016\pi}{6} = 0$.

Another way would be to directly compute that $\left(\sqrt{3}+i\right)^3 = 8i$, which gives $\left(\sqrt{3}+i\right)^6 = -64$, and finally $\left(\sqrt{3}+i\right)^{2016} = (-64)^{336}$ is real. □

11. (4 points) For which $x \in \mathbb{R}$ the vectors $a = (1, 0, 1)$, $b = (1, x, 2)$, $c = (3, -2, x)$ of the three dimensional space

   (a) are coplanar,

(b) form a right hand oriented basis.

Answer: (a) $x = 1$ *or* $x = 2$*, (b)* $x \in (-\infty, 1) \cup (2, +\infty)$.

Proof. Consider the determinant

$$D = \begin{vmatrix} 1 & 0 & 1 \\ 1 & x & 2 \\ 3 & -2 & x \end{vmatrix}.$$

The vectors a, b, c are coplanar if and only if $D = 0$. They form a right hand oriented basis if and only if $D > 0$. Since $D = x^2 - 3x + 2 = (x-2)(x-1)$, the result follows. □

12. (4 points) Let $T : U \to V$ be an isomorphism between vector spaces U and V over some field F. Prove that the map $T^{-1} : V \to U$ is linear.

Solution. We need to prove that for any vectors $v_1, v_2 \in V$ and any scalars $\alpha, \beta \in F$, $T^{-1}(\alpha v_1 + \beta v_2) = \alpha T^{-1}(v_1) + \beta T^{-1}(v_2)$.

Let $v_1, v_2 \in V$ and $\alpha, \beta \in F$. Let $u_1 = T^{-1}(v_1) \in U$ and $u_2 = T^{-1}(v_2) \in U$. Then

$$\alpha T^{-1}(v_1) + \beta T^{-1}(v_2) = \alpha u_1 + \beta u_2 = T^{-1}\left(T(\alpha u_1 + \beta u_2)\right).$$

Since T is linear, $T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2) = \alpha v_1 + \beta v_2$. Substitution into the above equation gives $\alpha T^{-1}(v_1) + \beta T^{-1}(v_2) = T^{-1}(\alpha v_1 + \beta v_2)$. □

13. (3 points) Consider the set of m-by-n matrices with complex entries, i.e., $M_{m,n}(\mathbb{C})$. It is a vector space over the field of complex numbers. What is its dimension $\dim(M_{m,n}(\mathbb{C}))$? Justify your answer.

Answer: mn.

Solution. For $1 \le i \le m$ and $1 \le j \le n$, consider the matrices A_{ij} with 1 on the intersection of the ith row and jth column and 0's everywhere else.

These matrices are linearly independent. Indeed, if $\alpha_{ij} \in \mathbb{C}$ and $\sum_{i=1}^m \sum_{j=1}^n \alpha_{ij} A_{ij} = 0_{m,n}$, where $0_{m,n}$ is an m-by-n matrix with all the entries equal to 0, then necessarily all α_{ij}'s are 0's.

These matrices are also complete in $M_{m,n}(\mathbb{C})$. Indeed, any matrix with entries α_{ij} equals to $\sum_{i=1}^m \sum_{j=1}^n \alpha_{ij} A_{ij}$.

Thus, A_{ij} form a basis of $M_{m,n}(\mathbb{C})$. Since the dimension of $M_{m,n}(\mathbb{C})$ is the number of elements in a basis, the result follows. □