EXAM SOLUTIONS, 19 February 2015, 10:00 – 12:00

1. (4 points) Prove the inequality $(2n)! \geq 2^n (n!)^2$ for all integers $n \geq 1$.

   Solution. The proof is by induction on n. For $n = 1$, $(2 \cdot 1)! = 2$ and $2^1(1!)^2 = 2$, thus the inequality holds. Assume that for some $n \geq 1$, $(2n)! \geq 2^n(n!)^2$, and prove that $(2(n+1))! \geq 2^{n+1}((n+1)!)^2$. Indeed, $(2(n+1))! = (2n)!(2n+1)(2n+2) \geq 2^n(n!)^2(2n+1)(2n+2) \geq 2^n(n!)^2(n+1)2(n+1) = 2^{n+1}((n+1)!)^2$. □

2. (4 points) Prove that there exists $x \in [0, 2]$ such that $e^x = \pi$.

   Solution. Note that e^x is continuous on $[0, 2]$, $e^0 = 1$, $e^2 > 2^2 = 4$, and $\pi \in [1, 4]$. Thus, by the intermediate value theorem, there exists $x \in [0, 2]$ such that $e^x = \pi$. □

3. (4 points) For which values of $a \in \mathbb{R}$ the following function is differentiable at 0?

$$f(x) = \begin{cases} x^a \sin \frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

   Solution. By definition, f is differentiable at 0 if there exists finite limit $\lim_{x \to 0} \frac{f(x)-f(0)}{x-0}$. We compute

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^a \sin \frac{1}{x}}{x} = \lim_{x \to 0} x^{a-1} \sin \frac{1}{x}.$$

   Noting that $|\sin \frac{1}{x}| \leq 1$ for all $x \neq 0$, the above limit equals to 0 for $a > 1$. On the other hand, if $a \leq 1$, then the above limit does not exist. One can see this, for instance, by taking limits along subsequences $x_n = \frac{1}{2\pi n}$ and $y_n = \frac{1}{\frac{\pi}{2} + 2\pi n}$. Thus, f is differentiable if and only if $a > 1$. □

4. (4 points) Let a be a positive real number. Prove that the function $f(x) = \frac{e^x}{x^a}$ is monotone increasing on the interval $[a, +\infty)$.

   Proof. Since f is differentiable on $(0, +\infty)$, it is monotone increasing on $[a, +\infty)$ if $f'(x) \geq 0$ for all $x \in (a, +\infty)$. We compute

$$f'(x) = \frac{e^x x^a - a x^{a-1} e^x}{x^{2a}} = \frac{e^x x^{a-1}(x - a)}{x^{2a}},$$

   which is non-negative for all $x \geq a > 0$. The proof is complete. □

5. (4 points) Compute the limit

$$\lim_{x \to 0} (\cos x)^{-\frac{1}{x^2}}.$$

Solution.
$$\lim_{x\to 0}(\cos x)^{-\frac{1}{x^2}} = \lim_{x\to 0} e^{-\frac{1}{x^2}\ln\cos x} = e^{-\lim_{x\to 0}\frac{1}{x^2}\ln\cos x},$$
where the second equality is because of continuity of the exponential function. Since $\lim_{x\to 0}\frac{1}{x^2} = \infty$, $\lim_{x\to 0}\ln\cos x = \infty$, and the limit of the ratio of derivatives exists and equals
$$\lim_{x\to 0}\frac{(\ln\cos x)'}{(x^2)'} = -\lim_{x\to 0}\frac{\sin x}{2x\cos x} = -\frac{1}{2}\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{1}{\cos x} = -\frac{1}{2},$$
we conclude from l'Hopital's theorem that $\lim_{x\to 0}\frac{1}{x^2}\ln\cos x = -\frac{1}{2}$, which gives us $\lim_{x\to 0}(\cos x)^{-\frac{1}{x^2}} = e^{\frac{1}{2}}$. □

6. Compute the following indefinite integrals:
$$\text{(a) (4 points)} \int \frac{dx}{x\ln^2 x}; \qquad \text{(b) (4 points)} \int x\ln x dx.$$

Solution. (a) We use substitution $u = \ln x$, $du = \frac{dx}{x}$. Then $\int \frac{dx}{x\ln^2 x} = \int \frac{du}{u^2} = -u^{-1} + C = -(\ln x)^{-1} + C$.

(b) We use integration by parts formula $\int udv = uv - \int vdu$ with $u = \ln x$ and $dv = xdx$. Then $du = \frac{dx}{x}$ and $v = \frac{x^2}{2}$, and we conclude that $\int x\ln x dx = \frac{x^2\ln x}{2} - \frac{1}{2}\int xdx = \frac{x^2\ln x}{2} - \frac{x^2}{4} + C$. □

7. (4 points) For which values of $a \in \mathbb{R}$ the following series converges?
$$\sum_{n=1}^{+\infty}\frac{n^a}{n!}.$$

Solution. By the ratio test, the series $\sum_{n=1}^{+\infty} a_n$ with all $a_n > 0$ converges if $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} < 1$. In our example, $a_n = \frac{n^a}{n!} > 0$, and $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)^a}{(n+1)!}\frac{n!}{n^a} = \lim_{n\to\infty}\frac{(n+1)^a}{n^a}\lim_{n\to\infty}\frac{1}{n+1} = 1\cdot 0 = 0 < 1$. Thus, by the ratio test, the series converges for any $a \in \mathbb{R}$. □

8. (4 points) Identify the radius of convergence of the following series:
$$\sum_{n=1}^{+\infty}\left(1-\frac{1}{n}\right)^{n^2} z^n \quad (z \in \mathbb{C}).$$

Solution. By the Cauchy-Hadamard formula, the radius of convergence of the power series $\sum_{n=1}^{+\infty} a_n z^n$ equals $\left(\limsup_{n\to\infty}\sqrt[n]{|a_n|}\right)^{-1}$. Applying this formula with $a_n = \left(1-\frac{1}{n}\right)^{n^2}$ and noting that $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^{-1}$, we conclude that the radius of convergence of the above series equals e. □

9. (4 points) Find the algebraic form of $\left(\frac{3+i}{1-i}\right)^2$.

Solution.

$$\left(\frac{3+i}{1-i}\right)^2 = \left(\frac{(3+i)(1+i)}{(1-i)(1+i)}\right)^2 = \left(\frac{2+4i}{2}\right)^2 = (1+2i)^2 = -3+4i.$$

□

10. (4 points) Find all the solutions to the equation $z^4 = \sqrt{3} + i$ in the form $z = re^{i\theta}$, where $r > 0$ and $\theta \in [0, 2\pi)$.

Solution. First we compute $\sqrt{3} + i = 2(\frac{\sqrt{3}}{2} + \frac{1}{2}i) = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = 2e^{i\frac{\pi}{6}}$. Let $z = re^{i\theta}$ be a solution to $z^4 = \sqrt{3} + i$. Then $r = \sqrt[4]{2}$ and $4\theta = \frac{\pi}{6} + 2\pi k$, $k \in \mathbb{Z}$. Among all the θ's, only those corresponding to $k = 0, 1, 2, 3$ are in $[0, 2\pi)$. Therefore, the solutions are $z = re^{i\theta}$ with $r = \sqrt[4]{2}$ and $\theta = \frac{\pi}{24}, \frac{\pi}{24} + \frac{\pi}{2}, \frac{\pi}{24} + \pi, \frac{\pi}{24} + \frac{3\pi}{2}$. □

11. (6 points) Which of the following 4 sets of vectors in $\mathbb{R}^3$ are linearly independent, which are linearly dependent, which form a basis of $\mathbb{R}^3$? Justify your answers.

(a) $(2, 1, 0)$, $(4, 2, 0)$;
(b) $(2, 1, 0)$, $(3, 2, 0)$;
(c) $(2, 1, 0)$, $(3, 2, 0)$, $(1, 2, 1)$;
(d) $(2, 1, 0)$, $(3, 2, 0)$, $(1, 2, 1)$, $(0, 0, 1)$.

Solution. (a) These vectors are collinear, thus linearly dependent.

(b) These vectors are non-collinear, thus linearly independent.

(c) Since the determinant of the matrix formed by the coordinates of these vectors is non-zero

$$\begin{vmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 4 - 3 = 1 \neq 0,$$

the vectors are linearly independent.

(d) Any 4 vectors in $\mathbb{R}^3$ are linearly dependent, thus these vectors are linearly dependent.

Since the dimension of $\mathbb{R}^3$ is 3, any basis of $\mathbb{R}^3$ consists of 3 vectors. Thus, the vectors in (a), (b), and (d) cannot form a basis of $\mathbb{R}^3$. The vectors in (c) are linearly independent, and there are 3 of them, thus they form a basis of $\mathbb{R}^3$. □

12. (3 points) Give the definition of isomorphism between two vector spaces U and V over a field F.

13. (3 points) Give the definition of rank of a linear map.

14. (3 points) Give the definition of characteristic polynomial of a linear operator on a finite dimensional vector space.